DOKTORSAVHANDLINGAR
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CHALMERS TEKNISKA HÖGSKOLA
Nr 54

PHOTON GEOMETRY, INTERNAL COORDINATES AND HIGHER SYMMETRIES FOR ELEMENTARY PARTICLES

BY

ARNE KIHLBERG

GÖTEBORG 1966

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Tekn. lic.

AKADEMISK AVHANDLING

SOM MED TILLSTÅND AV SEKTIONEN FÖR TEKNISK FYSIK VID CHALMERS TEKNISKA HÖGSKOLA FRAMLÄGGES TILL OFFENTLIG GRANSKNING FÖR TEKNOLOGIE DOKTORSGRADS VINNANDE ONSDAGEN DEN 11 MAJ 1966 KL. 10 Å HÖRSALEN I ADMINISTRATIONSBYGGNADEN VID CHALMERS TEKNISKA HÖGSKOLA, SVEN HULTINS GATA, GÖTEBORG

GÖTEBORG
ELANDERS BOKTRYCKERI AKTIEBOLAG
1966

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The present paper is an introduction and summary of a thesis comprising the following five papers

I. Kihlberg, A., On the internal degrees of freedom of elementary particles, Arkiv Fysik *28*, 121 (1964).

II. Kihlberg, A., On a class of explicit representations of the homogeneous Lorentz group, Arkiv Fysik *27*, 373 (1964).

III. Kihlberg, A., On the unitary representations of a class of pseudo-orthogonal groups, Arkiv Fysik *30*, 121 (1965).

IV. Kihlberg, A., On the unitary irreducible representations of the pseudo-orthogonal group L (3,3), Arkiv Fysik (1966).

V. Kihlberg, A., Some non-compact symmetry groups for elementary particles associated with a geometrical model, Arkiv Fysik (1966).

Symmetry plays an important role in physics. Symmetry means invariance of certain properties under substitutions. The substitutions need for their definition objects which can be substituted. Very often these objects are coordinates of one or several particles. In particular this is so for the Poincaré group P (the inhomogeneous Lorentz group), which is defined as a group of substitutions on the 4-dimensional coordinate system. Another symmetry is expressed as the permutation symmetry among indistinguishable particles. This latter symmetry leads to a discrete group of transformations while the former type in general leads to continuous groups, in fact Lie groups. The implication of a symmetry group is more involved in quantum mechanics than in classical mechanics. In quantum mechanics the presence of a symmetry group implies that the Hilbert space is a representation space for, generally, a unitary representation of the group.

Certain symmetry groups such as P or the permutation group $S(n)$ seem to be firmly established in elementary particle physics. For others the situation is less clear. The isospin group $SU(2)$ is a good symmetry group for strong interactions. Since it is a continuous Lie group one should like to be able to interpret it as a transformation group on some coordinate space. For $SU(3)$ the situation is even worse. Not only is the interpretation as a transformation group lacking but its predictions are not too convincing.

The last two years have seen a very high activity in the search for higher symmetry groups. One general idea has been to unify the Poincaré group and the internal symmetry group into a larger global group. Some authors have aimed at a complete description, i.e. not only of the kinematics but also of the dynamics. Thus also the mass spectra should emerge from such a scheme. Others, being more restrictive, have attempted to put the spin and the internal degrees of freedom on an equal footing in order to obtain an enlarged internal symmetry group with more predictive power than $SU(3)$.

It is in the light of this development the present thesis could be viewed although its main ideas were conceived before the advent of

$SU(6)$ and its relativistic imbeddings. We aim at a construction of a global group G which contains P and some other subgroup S which can be interpreted as the internal symmetry group replacing $SU(3)$. However, we consider it important not only to have a group but also to know what it acts on. Therefore, we do not prescribe S but instead we try to define a generalized relativistic coordinate space and then we look for transformation groups on this space. In this way we do not arrive at $SU(3)$ as the internal symmetry group. Instead we discuss three alternatives for S namely $SO(2)$, the rotation group in two dimensions, $L(1,3)$ the homogeneous Lorentz group and $L(3,3)$ the pseudoorthogonal group in three space and three time dimensions.

The underlying space should be operationally defined and at the same time closely connected to the Minkowsky space. By utilizing the spatial properties as well as the polarization properties of the photon it is in fact possible to define operationally not only the Minkowsky space but also to define at each point $(\mathbf{x}, t)$ the orientation of a six-vector $(\mathbf{e}_0, \mathbf{h}_0)$ and its length $|\mathbf{e}_0| = |\mathbf{h}_0|$. The vectors $\mathbf{e}_0$ and $\mathbf{h}_0$ are essentially the electric and magnetic field strengths of the photon and satisfy $\mathbf{e}_0 \cdot \mathbf{h}_0 = 0$. Since now directions can be measured strictly at a point we assume that an elementary particle has coordinates $(\mathbf{e}, \mathbf{h})$ which should be measured relative to $(\mathbf{e}_0, \mathbf{h}_0)$ at the point $(\mathbf{x}, t)$. The internal coordinate space thus has four dimensions. In this way we have enlarged the configuration space from the four-dimensional Minkowsky space to an eight-dimensional space. This construction is carried out in paper *I*. Furthermore, using as a guiding principle that transformations on the reference system, i.e. transformations on $(\mathbf{x}, t)$ and $(\mathbf{e}_0, \mathbf{h}_0)$ which leave their definition invariant could be symmetry transformations, we then construct a transformation group isomorphic to

$$G_2 = [\bar{P} \otimes \bar{L}(1,3)]/Z_2$$

where Z_2 denotes a discrete centre. In addition to this group we also study the groups

$$G_1 = [\bar{P} \otimes \overline{SO}(2)]/Z_2$$

$$G_3 = [\bar{P} \otimes \bar{L}(3,3)]/Z_3$$

which contain internal transformations, i.e. operations on the axis $(\mathbf{e}, \mathbf{h})$ of the particle itself.

In order to study the consequences for an elementary particle theory of the assumption that one of the groups G_i is a symmetry group we need their unitary irreducible representations. In paper III a general technique for finding these is developed. The method given there is of value by itself since it allows us in principle to construct the unitary representations of a large class of semi-simple noncompact groups. Paper IV is devoted to the derivation of a number of series of unitary irreducible representations of $L(3,3)$ by the method of paper III. Similarly, paper II contains the derivation of the unitary irreducible representations of $L(1,3)$ by essentially the same method. This paper was actually written before paper III and was in fact the preamble of paper III.

The application of the groups G_i to elementary particle physics is carried through in paper V. Since the groups are homomorphic to direct products of P and $SO(2)$, $L(1,3)$ and $L(3,3)$, respectively, we cannot expect to obtain mass spectra within the groups. Rather we are limited to a less ambitious program. Only the group G_3 is large enough to allow for an interpretation of isospin and possibly of hypercharge and baryon number. Thus we concentrate our attention to G_3 and examine whether the spectra of spin, isospin, hypercharge and baryon number resemble those found empirically. Although the number of particles and resonances whose quantum numbers are known with certainty is not very abundant we think it is fair to say that some of the spectra given by the group $L(3,3)$ coincide so well with the experimentally measured ones that one is tempted to go one step further and evaluate, say, consequences for scattering processes. It may be worth while to do this but it should be borne in mind that it is not so clear that G_3 can be considered as a true symmetry group of the interaction between particles since it contains transformations which do not act on the external frame of reference. In fact the most immediate generalization of G_3 to a many particle situation would imply, if taken as a symmetry group, separate isospin conservation for each individual particle. Therefore, for the moment we consider G_3 just as a spectrum generating group. The groups G_1 and G_2 on the other hand are good candidates for symmetry groups. While G_1 has little predictive power G_2 can be tested as a symmetry group. At least in our interpretation it does not, however, seem to give correct predictions.

Acknowledgements

To Professor Nils Svartholm, who during many years has guided and supported my scientific work, I owe a deeply felt gratitude. His advice and encouragement has not been limited to physics only.

To Laborator Jan Nilsson whose interest in my work has stimulated me I also owe sincere thanks.

Part of this thesis was written during a stay at Cern 1964—1965. I want to thank prof. L. van Hove for the hospitality extended to me there.

GÖTEBORG
ELANDERS BOKTRYCKERI AKTIEBOLAG
1966

54 Bilaga till Doktorsavhandl. Nr. 54

Tp

ARKIV FÖR FYSIK

UTGIVET AV

KUNGL. SVENSKA VETENSKAPSAKADEMIEN

Band 28 nr 12

ARNE KIHLBERG

On the internal degrees of freedom of elementary particles

ALMQVIST & WIKSELL

STOCKHOLM

GÖTEBORG · UPPSALA

1964

ARKIV FÖR FYSIK Band 28 nr 12

3.64 084 Communicated 13 May 1964 by I. Waller and T. Gustafson

On the internal degrees of freedom of elementary particles

By Arne Kihlberg

ABSTRACT

We examine the use of a six-vector $(\mathbf{e}, \mathbf{h})$, with $\mathbf{e} \cdot \mathbf{h} = 0$, $\mathbf{e}^2 = \mathbf{h}^2$, as internal coordinates for a particle in addition to its space time coordinates. Using the operational definition of $(\mathbf{e}, \mathbf{h})$, we try to find an appropriate invariance group which can replace the inhomogeneous Lorentz group. Different choices are conceivable, however, and we discuss in particular the case of a group which is isomorphic to the direct product of the inhomogeneous Lorentz group and the homogeneous Lorentz group. We also speculate about the connection between the quantum numbers of the elementary particles and the spectra of a unitary representation of the group.

1. Introduction

In recent years several authors [1, 2] have discussed the possibility of using internal coordinates for the description and classification of elementary particles. They all assume that a particle is some sort of rotator which should be analyzed in terms of a vector structure attached to the position coordinates. Vigier *et al.* [1] base their model on the theory of the relativistic fluid, while others [2] take as starting point the observed spectra of certain quantum numbers such as isospin and baryon number.

In this paper we try to motivate a special model geometrically, i.e., we define all coordinates in a way similar to the procedure used in relativity. Now relativity is based on a certain definition of time [3], in which the photon plays a decisive role. Since we have accepted the privileged position of the photon in relativity it seems to us that we should make use of its full capacity.

2. Definition of an eight-dimensional reference system

We shall define an eight-dimensional reference system of coordinates in a way analogous to the manner in which the four-dimensional reference system of space and time is defined in relativity. Assume that we have a radar station which is capable of emitting arbitrarily sharp, polarized pulses. After a certain elapse of time the pulse returns if it is reflected at some point. The station is further assumed to be located on some massive, rigid body so that the antenna can be turned and translated. We can then measure three directional angles of the antenna and two readings t_1 and t_2 of a clock at the emission and reabsorption of the pulse. Two of the angles and the difference $t_2 - t_1$ can be used to define relative cartesian coordinates (x_1, x_2, x_3) in an obvious way. The local time at the point (x_1, x_2, x_3) at the moment of reflection is obtained by means of Einstein's definition $t = \frac{1}{2}(t_1 + t_2)$. The third angle specifies the polarization of the signal in a plane orthogonal to the direction of emission and defines at the world point (x_1, x_2, x_3, t) a reference for a cyclic coordinate ψ in this plane. It is hard to imagine any more quantity which could be measured at such a radar station except possibly the intensity of the pulse. The frequency of the signal is necessarily undetermined since we use a sharp pulse.

According to our definition the coordinate ψ is fixed to the radar station. If we translate the station then ψ is referred to a different plane, while the local time is not affected by this operation. This leads us to consider all the three directional angles as independent coordinates. At every space point (x_1, x_2, x_3) there is thus a synchronized clock, showing the local time, and two orthogonal three-vectors $\mathbf{e}$ and $\mathbf{h}$ of equal length, which can be thought of as the electric and magnetic field vectors of the radar signal. The length of $\mathbf{e}$ and $\mathbf{h}$ is obviously related to the strength of the radar pulse and $\mathbf{e} \times \mathbf{h}$ is directed from the station.

The Einstein definition of the local time is one out of many possible choices. However, it guarantees that the velocity of light, is constant and that is the physical principle on which the theory of relativity is based. There is another principle, although it is not always explicitly stated, that the polarization of a photon does not change when it is sent from one point to another in empty space. Therefore, we choose to define our six-vectors $(\mathbf{e}, \mathbf{h})$ at different space-points parallel to each other. Just as the definition of time in relativity guarantees that the velocity of light is always c our definition of the vectors $(\mathbf{e}, \mathbf{h})$ guarantees that the direction of polarization of a photon is the same in all points it passes.

In this way we have arrived at a reference system containing eight parameters $(x_1, x_2, x_3, t, \mathbf{e}, \mathbf{h})$ where $(\mathbf{e}, \mathbf{h})$ is a six-vector satisfying $\mathbf{e} \cdot \mathbf{h} = 0$, $\mathbf{e}^2 = \mathbf{h}^2$. It is then natural to assume that a particle (at least for the classical case) possesses all these degrees of freedom, i.e., in total eight coordinates $(x_1, x_2, x_3, t, \mathbf{e}, \mathbf{h})$. The coordinates x_1, x_2, x_3 give the point in space where the particle is and t is the corresponding reading of the clock at (x_1, x_2, x_3). The vectors $(\mathbf{e}, \mathbf{h})$ are finally expressed in components on the standard directions at (x_1, x_2, x_3).

3. Connection with the classification of Finkelstein

Finkelstein [4] has classified all internal structures of particles within the framework of relativity. He proceeds in the following way. Assume that our theory is invariant under some group G (the inhomogenous Lorentz group, IHLG) and that the coordinates of the particle are called x. Then every group element $g \in G$ induces a transformation in the coordinate space M to which x belongs. Let us call this transformation $h(g)$,

$$x \to h(g)\,x.$$

If furthermore G acts transitively, i.e., every point in M can be reached from x by means of a transformation induced by a suitable g, then one can show that M can be identified with a coset space of G. To show this let G_x be that subgroup of G which leaves x invariant,

$$h(g)\,x = x \quad \text{for} \quad g \in G_x.$$

Furthermore let $g_1, g_2 \in G$ be such that

$$h(g_1)\,x = y = h(g_2)\,x,$$

then
$$h(g_2^{-1})h(g_1)x = x,$$
from which follows that
$$g \equiv g_2^{-1} g_1 \in G_x,$$
so that y can be identified with the set $\{g_2 g\}_{g \in G_x}$ which is the left coset belonging to the subgroup G_x. Because of transitivity all $y \in M$ can be written as $h(g)x$ for some $g \in G$.

Thus one can classify all coordinate structures by enumerating all coset spaces, i.e., by enumerating all inequivalent subgroups of G. When G is the IHLG, two of the most interesting coordinate structures are the one we discussed in the preceeding paragraph (eight-dimensional), and the one which is identical with the group space (ten-dimensional). The latter is used by Vigier *et al.* in their theory.

4. The symmetry group of the eight-dimensional space

As soon as the local time has been defined in relativity one can ask which coordinate transformations are compatible with the principle of constant velocity of light. In order to arrive at the IHLG one must restrict oneself to linear transformations or postulate that massive particles, moving on a straight line, should continue to do so after the coordinate transformation [3]. In both cases it is also necessary explicitly to exclude scale transformations.

In order to find the appropriate symmetry group of the eight-dimensional space we proceed in a similar way. First we look for those transformations which leave invariant the velocity of light and the parallellity of the six-vectors. It is obvious that this group of transformations includes the IHLG as a subgroup for if we look at the six-vectors $(\mathbf{e}, \mathbf{h})$ from a moving system they appear to be turned and their length has changed, but the vectors at different points are still parallel to each other. But we obviously have other possibilities. We can turn the six-vectors arbitrarily and scale their length, or we can turn them only around $\mathbf{e} \times \mathbf{h}$ and scale their length. How large the extension beyond the IHLG will be depends on the additional restrictions we impose on the transformations. For the moment we shall assume that all uniform rotations of the six-vectors are symmetry operations. If these transformations together with the transformations of the IHLG are to make up a group we must accept separate accelerations of the six-vectors as well. In this case we arrive at a sixteen-parameter group.

Let $\mathbf{S}$ and $\mathbf{S}'$ denote the generators of rotations and accelerations of the six-vectors along given direction in space. Furthermore let $(\mathbf{e}_0, \mathbf{h}_0)$ stand for the fixed reference orientation of the vectors $(\mathbf{e}, \mathbf{h})$. They are so to speak the origin of the variables $(\mathbf{e}, \mathbf{h})$. We then have the following expressions for $\mathbf{S}$ and $\mathbf{S}'$

$$\left.\begin{aligned} \mathbf{S} &= \mathbf{e}_0 \times \nabla_{e_0} + \mathbf{h}_0 \times \nabla_{h_0}, \\ \mathbf{S}' &= -\mathbf{h}_0 \times \nabla_{e_0} + \mathbf{e}_0 \times \nabla_{h_0}. \end{aligned}\right\} \tag{1}$$

From eqs. (1) we get the following commutation relations

$$\left.\begin{aligned} [S_1, S_2] &= -S_3 \quad \text{(cyclic)}, \\ [S_1', S_2'] &= S_3 \quad\;\;\, \text{(cyclic)}, \\ [S_1, S_2'] &= -S_3' \quad \text{(cyclic)}, \\ [S_1', S_2] &= -S_3' \quad \text{(cyclic)}, \end{aligned}\right\} \tag{2}$$

while all other commutators are zero. These relations are of course those of the homogeneous Lorentz group. However, the only change $(\mathbf{e}_0, \mathbf{h}_0)$ undergoes is rotation and scaling and therefore it must be possible to express $\mathbf{S}$ and $\mathbf{S}'$ in terms of the operators

$$\left.\begin{aligned} T_1 &= \hat{\mathbf{e}}_0 \cdot (\mathbf{h}_0 \times \nabla_{h_0}) \quad \hat{\mathbf{e}}_0 = \frac{\mathbf{e}_0}{|\mathbf{e}_0|}, \\ T_2 &= \hat{\mathbf{h}}_0 \cdot (\mathbf{e}_0 \times \nabla_{e_0}) \quad \hat{\mathbf{h}}_0 = \frac{\mathbf{h}_0}{|\mathbf{h}_0|}, \\ T_3 &= \mathbf{h}_0 \cdot \nabla_{e_0} - \mathbf{e}_0 \cdot \nabla_{h_0}, \\ D &= \mathbf{e}_0 \cdot \nabla_{e_0} + \mathbf{h}_0 \cdot \nabla_{h_0}, \end{aligned}\right\} \tag{3}$$

which are the generators of rotation around $\mathbf{e}_0, \mathbf{h}_0$ and $\mathbf{e}_0 \times \mathbf{h}_0$ and scaling of the length of $\mathbf{e}_0$ and $\mathbf{h}_0$. In fact we have

$$\mathbf{S} = \hat{\mathbf{e}}_0 T_1 + \hat{\mathbf{h}}_0 T_2 + \hat{\mathbf{e}}_0 \times \hat{\mathbf{h}}_0 T_3,$$

$$\mathbf{S}' = \hat{\mathbf{e}}_0 T_2 - \hat{\mathbf{h}}_0 T_1 + \hat{\mathbf{e}}_0 \times \hat{\mathbf{h}}_0 (D-1).$$

We get an alternative representation of the operators $\mathbf{S}$ and $\mathbf{S}'$ by introducing as independent coordinates the Euler angles φ, θ, ψ between the six-vector $(\mathbf{e}_0, \mathbf{h}_0)$ and the six-vector $(\mathbf{e}, \mathbf{h})$ of the particle, and the scale coordinate $s = \ln|\mathbf{e}|/|\mathbf{e}_0|$. Then T_1, T_2, T_3 and D take the form

$$\left.\begin{aligned} T_1 &= +\sin\psi \frac{\partial}{\partial\theta} + \cos\psi \cot\theta \frac{\partial}{\partial\psi} - \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\varphi}, \\ T_2 &= +\cos\psi \frac{\partial}{\partial\theta} - \sin\psi \cot\theta \frac{\partial}{\partial\psi} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\varphi}, \\ T_3 &= \frac{\partial}{\partial\psi}, \\ D &= -\frac{\partial}{\partial s}, \end{aligned}\right\} \tag{4}$$

while $\mathbf{S}$ and $\mathbf{S}'$ are

$$S_1 = -\sin\varphi \frac{\partial}{\partial\theta} - \cos\varphi \cot\theta \frac{\partial}{\partial\varphi} + \frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\psi},$$

$$S_2 = \cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} + \frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\psi},$$

$$S_3 = \frac{\partial}{\partial\varphi},$$

$$S_1' = +\cos\theta \cos\varphi \frac{\partial}{\partial\theta} + \sin\varphi \cot\theta \frac{\partial}{\partial\psi} - \frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi} + \sin\theta \cos\varphi \left(\frac{\partial}{\partial s} - 1\right),$$

$$S_2' = +\cos\theta\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\cot\theta\frac{\partial}{\partial\psi} + \frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi} + \sin\theta\sin\varphi\left(\frac{\partial}{\partial s} - 1\right),$$

$$S_3' = -\sin\theta\frac{\partial}{\partial\theta} + \cos\theta\left(\frac{\partial}{\partial s} - 1\right).$$

The invariants of the Lie algebra generated by $\mathbf{S}$ and $\mathbf{S}'$ are

$$\left.\begin{aligned} R &= \mathbf{S}^2 - \mathbf{S}'^2 = -T_3^2 + (D-1)^2 + 2(D-1), \\ S &= \mathbf{S}\cdot\mathbf{S}' = -T_3 D. \end{aligned}\right\} \tag{6}$$

The generators of the inhomogeneous transformations can be written in the form

$$\left.\begin{aligned} p_t &= \frac{\partial}{\partial t}, \\ p_i &= \frac{\partial}{\partial x_i} \quad (i = 1, 2, 3), \\ M_i &= M_i^0 + S_i, \\ N_i &= N_i^0 + S_i', \end{aligned}\right\} \tag{7}$$

where $$M_1^0 = x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2} \quad \text{(cyclic)},$$

$$N_i^0 = t\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial t}.$$

From eqs. (1) and (7) it is evident that the Lie algebra of our 16-parameter invariance group is the direct product of the Lie algebras generated by $(p_t, \mathbf{p}, \mathbf{M}^0, \mathbf{N}^0)$ and $(\mathbf{S}, \mathbf{S}')$. Thus the total number of invariants is four namely R, S and

$$P = \mathbf{p}^2 - p_t^p,$$

$$Q = (\mathbf{p}\cdot\mathbf{M}^0)^2 - (p_t\mathbf{M}^0 + \mathbf{p}\times\mathbf{N}^0)^2.$$

Q is identically zero because $\mathbf{M}^0$ and $\mathbf{N}^0$ operate on a four-dimensional space.

So far we have discussed a coordinate space and a symmetry group of transformations which connects different choices of axis in it. All physical laws should be invariant under this group if it is impossible to distinguish between the different coordinate systems by means of experiments. The states of a quantal system should transform according to a unitary representation of the group. Hopefully, the irreducible representations may be connected to the "elementary" quantal system or, in other words, the elementary particles.

We now assume that the elements of the Hilbert space, in which the representations operate, are functions of the coordinates. The hermitian generators of the representation are taken to be $\tilde{p}_t = -ip_t$, $\tilde{\mathbf{P}} = -i\mathbf{p}$, $\tilde{\mathbf{M}}^0 = -i\mathbf{M}^0$, $\tilde{\mathbf{N}}^0 = -i\mathbf{N}^0$, $\tilde{\mathbf{S}} = -i\mathbf{S}$ and $\tilde{\mathbf{S}}' = -i\mathbf{S}'$. These operators form a Lie algebra with the invariants $-P$, Q, $-R$ and $-S$. The invariants are fixed numbers in an irreducible representation. Since all symmetry operations are external in the sense that they operate on the frame of reference, quantum numbers characterizing a particle must be sought among these invariants. P is usually interpreted as the mass of the particle. R and S, or alternatively $\tilde{T}_3 = -\mathrm{i}T_3$ and $\tilde{D} = -\mathrm{i}D$, do not lend themselves to any immediate interpretation. Now the eigenvalue of $\tilde{T}_3$, which may be integer or half-integer, is the lowest l-value in the reduction of a representation into irreducible representations of the three-dimensional rotation group generated by $\mathbf{S}$ [5]. Thus a particle invariant under our group does not have a fixed but a lowest spin-value $\tilde{T}_3$. This interpretation presupposes that the spin is connected with rotations of the six-vector $(\mathbf{e}_0, \mathbf{h}_0)$. One might as well assume that the spin is something characterizing the system under the rotation of space coordinates. Then the spin is zero because Q is zero. In this case we are free to identify $\tilde{T}_3$ with the third component of isospin.

Discussion

On the basis of a generalized light geometry we have tried to motivate an additional set of coordinates for a particle. These coordinates are four in number and can be represented by a six-vector $(\mathbf{e}, \mathbf{h})$ where $\mathbf{e}\cdot\mathbf{h} = 0$ and $\mathbf{e}^2 = \mathbf{h}^2$. This extension of the coordinate space suggests an extension of the group under which the physical laws should be invariant. In this paper we have examined which group one obtains in case one assumes that, in addition to the transformations of the IHLG, one can also rotate the six-vectors arbitrarily. The resulting group is the direct product of the HLG and the IHLG. The analysis in the preceding section of the physical interpretation shows that this group introduces some unfamiliar features. However, we consider it more as an example of the many possibilities one has when searching for new invariance groups.

To mention another possibility one might choose a group generated by p_t, $\mathbf{p}$, $\mathbf{M}$, $\mathbf{N}$ and T_3. Then $\mathbf{S}$ is likely to be interpreted as the spin operator and $(\tilde{p}_t\tilde{\mathbf{M}} + \tilde{\mathbf{p}}\times\tilde{\mathbf{N}})^2 - (\tilde{\mathbf{p}}\cdot\tilde{\mathbf{M}})^2$ as the square of the spin times the mass. $\tilde{T}_3$ is again an invariant and is linked together with the spin since $\tilde{T}_3$ is an integer or half-integer when the spin is an integer or half-integer. Also the value of $\tilde{T}_3$ is less or equal to the value of the spin. This suggests that we interprete $\tilde{T}_3$ as half the baryonic number.

Another possibility is to widen the coordinate space and invariance group even more. This could be done by giving a particle a second mechanical, "rigid" structure. One could then introduce truly internal coordinates in the spirit of Vigier *et al.*, which could be defined through the relation between the six-vector and the mechanical structure. The internal invariance group may then be chosen as the bilateral rotation group [1] if it is assumed that the particle is "spherical". Since these truly internal coordinates are not accessible to external transformations the states of this internal group may define different particles. In the case of the bilateral rotation group one obtains three quantum numbers which could be identified with isospin, its third component and hypercharge.

ACKNOWLEDGEMENT

The author wants to express his gratitude to Professor N. Svartholm for his kind interest and for his criticism and suggestions. I also thank Dr. J. Nilsson for reading the manuscript critically.

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Tryckt den 10 november 1964

Uppsala 1964. Almqvist & Wiksells Boktryckeri AB

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Nr. 54

ARKIV FÖR FYSIK

UTGIVET AV
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Band 27 nr 27

ARNE KIHLBERG

On a class of explicit representations of the homogeneous Lorentz group

ALMQVIST & WIKSELL
STOCKHOLM
GÖTEBORG · UPPSALA
1964

ARKIV FÖR FYSIK Band 27 nr 27

3.64 08 5 Communicated 13 May 1964 by I. Waller and T. Gustafson

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By Arne Kihlberg

ABSTRACT

A class of explicit, irreducible, unitary representations of the homogeneous Lorentz group is given. It is explicit in the sense that the Hilbert space is a certain function space over a carrier space. As the carrier space we choose the four-dimensional space of restricted six-vectors $(\mathbf{e}, \mathbf{h} \mid \mathbf{e} \cdot \mathbf{h} = 0,\ \mathbf{e}^2 = \mathbf{h}^2)$. The class is complete in the sense that any irreducible, unitary representation is unitarily equivalent to one in the class.

1. Introduction

The unitary representations up to unitary equivalence of the homogeneous Lorentz group (HLG) are all well known [1]. They can be completely reduced into irreducible, unitary representations. These are divided into two series. The main series contains representations which are characterized by two numbers (k_0, c), where k_0 is a non-negative integer or half-integer and c is real. The representations of the supplementary series are labelled by one number σ, $(\sigma = 2ic)$ where $0 < \sigma \leqslant 2$. These latter do not appear in the reduction of the regular representation into irreducible constituents.

Neumark [1] uses two different function spaces to realize the irreducible representations. They are connected to the group itself because the carrier spaces, on which the functions are defined, are subgroups of the HLG. The general group element induces a transformation in these subgroups according to a certain prescription. With a scalar product, suitably defined, the function spaces become Hilbert spaces and the transformations induced by the group are represented by unitary operators acting on these Hilbert spaces. It turns out, however, that different scalar products are required for the main series and the supplementary series.

In this paper we discuss the realization of the irreducible representations on a function space which is related to one of the two used by Neumark [1]. Our carrier space is the space of six-vectors $(\mathbf{e}, \mathbf{h})$ with $\mathbf{e} \cdot \mathbf{h} = 0$, $\mathbf{e}^2 = \mathbf{h}^2$, and thus four-dimensional. The HLG induces transformations in this carrier space and consequently in the function space defined on it. The invariants of the HLG have a simple meaning in this space. Again one has to define different scalar products to get the unitary representations of the main series and the supplementary series.

The reason for writing the representations in this explicit form is physical. We know that the treatment of interaction can have a simple formulation in one representation but not in another, which is mathematically equivalent. The best known example of this is the electro-magnetic field which in absence of interaction can equally well be described by means of the field tensor $F^{\mu\nu}$ and the four-vector A^{μ}. Both theories define a representation of the HLG, but the formulation involving A^{μ} seems to be more suited for treating the interaction. Quite apart from the question of interaction one explicit representation may be more illuminating and suggestive than another for the interpretation of the various mathematical expresssions which appear. Elsewhere [2], we have given arguments for using the restricted six-vector $(\mathbf{e}, \mathbf{h})$ as internal coordinates for a particle and one can make the hypothesis that the carrier space for the representation should coincide with the coordinate space of the physical system transforming according to this representation.

To construct the representations we apply the theory of multiplier representations described by Bargmann [3].

2. Definition of carrier space, function space and the transformations induced in them by the HLG

Our carrier space is the restricted six-vector $F \equiv (\mathbf{e}, \mathbf{h} \,|\, \mathbf{e}\cdot\mathbf{h} = 0,\ \mathbf{e}^2 = \mathbf{h}^2)$. As coordinates we can use either the six components $e_x, \ldots, h_z$ with the restrictions $e_x h_x + e_y h_y + e_z h_z = 0,\ e_x^2 + e_y^2 + e_z^2 = h_x^2 + h_y^2 + h_z^2$, or four independent coordinates $(\varphi, \theta, \psi, s)$. These we define through the equations

$$\left.\begin{aligned}
\begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} &= e^s \begin{bmatrix} -\sin\varphi\sin\psi + \cos\theta\cos\varphi\cos\psi \\ \cos\varphi\sin\psi + \cos\theta\sin\varphi\cos\psi \\ \sin\theta\cos\psi \end{bmatrix} \\
\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} &= e^s \begin{bmatrix} -\sin\varphi\cos\psi - \cos\theta\cos\varphi\sin\psi \\ \cos\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi \\ \sin\theta\sin\psi \end{bmatrix}
\end{aligned}\right\} \tag{1}$$

For completeness we also introduce

$$\begin{bmatrix} (\mathbf{e}\times\mathbf{h})_x \\ (\mathbf{e}\times\mathbf{h})_y \\ (\mathbf{e}\times\mathbf{h})_z \end{bmatrix} = e^{2s} \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} \tag{2}$$

From these equations we see that $s = \ln|\mathbf{e}|$ and that φ, θ, ψ are the Euler angles orienting the triad defined by $\mathbf{e}, \mathbf{h}$ and $\mathbf{e}\times\mathbf{h}$ with respect to some fixed reference frame.

As the function space we shall take a suitable subset of the set of all functions (real or complex) of $(\varphi, \theta, \psi, s)$ or of F. In the latter case the functions are only defined on the hypersurface $\mathbf{e}\cdot\mathbf{h}=0$, $\mathbf{e}^2=\mathbf{h}^2$. The subset must be chosen so that the function space is a linear space and so that it becomes a Hilbert space by defining a scalar product. We return to these questions later.

Let us now consider a restricted homogeneous Lorentz transformation defined by the matrix Ω^{μ}_{ν} ($\Omega^0_0>0$, det $\Omega=+1$). We define the action on F by the equation

$$F^{\mu\nu}\xrightarrow{\Omega}F'^{\mu\nu}=\Omega^{\mu}_{\varrho}\,\Omega^{\nu}_{\sigma}\,F^{\varrho\sigma}, \tag{3}$$

where
$$F^{\mu\nu}=\begin{bmatrix} 0 & e_x & e_y & e_z \\ -e_x & 0 & h_z & -h_y \\ -e_y & -h_z & 0 & h_x \\ -e_z & h_y & -h_x & 0 \end{bmatrix} \tag{4}$$

in analogy with the transformation properties of the electromagnetic field tensor. The corresponding transformation in the function space we shall define in the following way

$$f(F^{\mu\nu})\xrightarrow{\Omega}f'(F^{\mu\nu})=f(\Omega_{\varrho}^{-1\mu}\,\Omega_{\sigma}^{-1\nu}\,F^{\varrho\sigma}). \tag{5}$$

For an infinitesimal transformation we can write

$$\Omega^{\mu}_{\varrho}=\delta^{\mu}_{\varrho}+\omega^{\mu}_{\varrho}, \tag{6}$$

where
$$g_{\mu\sigma}\,\omega^{\mu}_{\varrho}+g_{\mu\varrho}\,\omega^{\mu}_{\sigma}=0. \tag{7}$$

Eq. 5 then becomes

$$f(F^{\mu\nu})\to f'(F^{\mu\nu})=f(F^{\mu\nu}-F^{\mu\sigma}\,\omega^{\nu}_{\sigma}-F^{\varrho\nu}\,\omega^{\mu}_{\varrho}). \tag{8}$$

If we examine separately space rotations and accelerations and write eq. 8

$$\left.\begin{aligned} f'(F)&=(1+\boldsymbol{\omega}\cdot\mathbf{S})\,f(F),\\ f'(F)&=(1+\boldsymbol{\omega}'\cdot\mathbf{S}')\,f(F)\end{aligned}\right\} \tag{9}$$

we then get the following expressions for the generators $\mathbf{S}$ and $\mathbf{S}'$ assuming that the function f is differentiable.

$$\left.\begin{aligned} \mathbf{S}&=\mathbf{e}\times\nabla_e+\mathbf{h}\times\nabla_h,\\ \mathbf{S}'&=\mathbf{e}\times\nabla_h-\mathbf{h}\times\nabla_e.\end{aligned}\right\} \tag{10}$$

The commutation relations between the different components of $\mathbf{S}$ and $\mathbf{S}'$ are given in ref. 2. If we use the second set of variables (φ,θ,ψ,s) we have

$$\left.\begin{aligned}
S_1 &= -\sin\varphi\,\frac{\partial}{\partial\theta} - \cos\varphi\,\cot\theta\,\frac{\partial}{\partial\varphi} + \frac{\cos\varphi}{\sin\theta}\,\frac{\partial}{\partial\psi},\\
S_2 &= \cos\varphi\,\frac{\partial}{\partial\theta} - \sin\varphi\,\cot\theta\,\frac{\partial}{\partial\varphi} + \frac{\sin\varphi}{\sin\theta}\,\frac{\partial}{\partial\psi},\\
S_3 &= \frac{\partial}{\partial\varphi},\\
S_1^1 &= \cos\theta\,\cos\varphi\,\frac{\partial}{\partial\theta} + \sin\varphi\,\cot\theta\,\frac{\partial}{\partial\psi} - \frac{\sin\varphi}{\sin\theta}\,\frac{\partial}{\partial\varphi} + \sin\theta\,\cos\varphi\left(\frac{\partial}{\partial s} - 1\right),\\
S_2^1 &= \cos\theta\,\sin\varphi\,\frac{\partial}{\partial\theta} - \cos\varphi\,\cot\theta\,\frac{\partial}{\partial\psi} + \frac{\cos\varphi}{\sin\theta}\,\frac{\partial}{\partial\varphi} + \sin\theta\,\sin\varphi\left(\frac{\partial}{\partial s} - 1\right).\\
S_3^1 &= -\sin\theta\,\frac{\partial}{\partial\theta} + \cos\theta\left(\frac{\partial}{\partial s} - 1\right).
\end{aligned}\right\} \tag{11}$$

3. The Casimir operators and the irreducibility condition

We have decided to use functions $f(\varphi, \theta, \psi, s)$ in the representation space. Now we require the space to be irreducible. Such a restriction is obtained if all functions are eigen-functions of the Casimir operators of the Lie-algebra (11). It is well known that these are

$$\left.\begin{aligned}
R &= \mathbf{S}^2 - \mathbf{S}'^2,\\
Q &= \mathbf{S}\cdot\mathbf{S}'.
\end{aligned}\right\} \tag{12}$$

In terms of our variables we get

$$\left.\begin{aligned}
R &= \frac{\partial^2}{\partial\psi^2} - \frac{\partial^2}{\partial s^2} + 1,\\
Q &= \frac{\partial^2}{\partial\psi\,\partial s}.
\end{aligned}\right\} \tag{13}$$

Thus all functions f in our representation space should satisfy the equations

$$\left.\begin{aligned}
\frac{\partial^2 f}{\partial\psi^2} - \frac{\partial^2 f}{\partial s^2} &= (r-1)\,f,\\
\frac{\partial^2 f}{\partial\psi\,\partial s} &= qf
\end{aligned}\right\} \tag{14}$$

for suitable real r and q. The restriction of r and q to real values is motivated by the fact that we later on introduce a unitarity condition. Now since we consider both double valued and single valued representations, f must be periodic in ψ with the period 4π and we may expand it in a Fourier series

$$f=\sum_{m=0}^{\infty}\left[A_m(s)\sin\frac{m\psi}{2}+B_m(s)\cos\frac{m\psi}{2}\right]. \tag{15}$$

The functions A_m and B_m must then satisfy the following equations

$$\left.\begin{aligned}&\frac{d^2A_m}{ds^2}+\left(r+\frac{m^2}{4}-1\right)A_m=0,\\ &\frac{d^2B_m}{ds^2}+\left(r+\frac{m^2}{4}-1\right)B_m=0,\\ &\frac{m}{2}\frac{dA_m}{ds}=qB_m,\\ &-\frac{m}{2}\frac{dB_m}{ds}=qA_m.\end{aligned}\right\} \tag{16}$$

We see that each equation contains only one m-value so that we may use m for the enumeration of the different possible solutions.

(I) Suppose $m\neq 0$.

Then
$$A_m=C_1^m\sin\nu s-C_2^m\cos\nu s,$$
$$B_m=C_2^m\sin\nu s+C_1^m\cos\nu s,$$

where
$$\nu=\frac{2q}{m},\ -\infty<\nu<\infty,$$

so that r and q can take on the values

$$r=1-\frac{m^2}{4}+\nu^2,$$

$$q=\frac{m\nu}{2}.$$

(II) $m=0$.

Then only $B_0(s)$ remains to be determined and we have three possibilities

(a)
$$B_0=C_2^0\sin\nu s+C_1^0\cos\nu s$$
$$-\infty<\nu<\infty$$

in which case
$$r=1+\nu^2,$$

$$q=0;$$

(b)
$$B_0 = D_1 e^{-\alpha s} + D_2 e^{\alpha s}$$
$$-\infty < \alpha < \infty$$

with
$$r = 1 - \alpha^2,$$
$$q = 0:$$

(c)
$$B_0 = E_1 + E_2 s$$

n which case
$$r = 1,$$
$$q = 0.$$

The case (II a) can evidently be incorporated under (I).

We thus arrive at the following conclusion: The irreducible function spaces can be enumerated by two numbers (m, ν). For m integer and ν real, the functions depend on ψ and s as

$$f(\varphi, \theta, \psi, s) = g(\theta, \varphi) \sin\left(\frac{m}{2}\psi - \nu s\right) + h(\theta, \varphi) \cos\left(\frac{m}{2}\psi - \nu s\right). \tag{17}$$

The operators (11) in the Lie-algebra mix the "components" g and h and it is obviously convenient to introduce the imaginary unit i and write

$$f = \tfrac{1}{2}(h - ig) \exp\left(\frac{m}{2}\psi - \nu s\right) + \tfrac{1}{2}(h + ig) \exp\left(-\frac{m}{2}\psi + \nu s\right). \tag{18}$$

In this way the real space is split into two complex conjugate spaces which are separately invariant for the Lie-algebra. When $m = 0$ both parts are covered by letting ν take both positive and negative values.

For $m = 0$ and ν imaginary but different from zero $(\nu = i\alpha)$ the functions depend on s in the following way

$$f = g(\theta, \varphi) e^{-\alpha s} + h(\theta, \varphi) e^{\alpha s}. \tag{19}$$

Now the operators (13) do not mix the components g and h and we can keep the real representation space. Again we have two separately invariant parts and by letting α take both positive and negative values we may just consider the functions

$$f = g(\theta, \varphi) e^{-\alpha s}. \tag{20}$$

Finally for $m = 0$ and $\nu = 0$ there is also the possibility

$$f = g(\theta, \varphi) s + h(\theta, \varphi). \tag{21}$$

4. The unitarity condition

In the preceding paragraph we have arrived at restrictions on the numbers (m, ν), which are used to define the irreducible function spaces, from reality conditions. The result was that either m is a non-negative integer and ν real (this is going to become the main series of the unitary representations) or m zero and ν imaginary but unequal to zero (part of this will constitute the supplementary series of unitary representations). If both m and ν are zero there is a further possibility, but it will turn out that it can be considered as a limiting case of the main series.

Now we want to see whether unitarity imposes any conditions on m and ν. Unitarity means that $\mathbf{S}$ and $\mathbf{S}'$ are anti-hermitian. Of course, we must then suppose that we have defined a scalar product. Then

$$\left.\begin{aligned}(f_1, \mathbf{S}\, f_2) &= -(\mathbf{S} f_1,\ f_2),\\ (f_1, \mathbf{S}' f_2) &= -(\mathbf{S}' f_1, f_2),\end{aligned}\right\} \tag{22}$$

for such f_1, f_2 which belong to the domain of $\mathbf{S}$ and $\mathbf{S}'$.

For the main series, i.e., ν real and

$$f = \exp\left(\frac{m}{2}\psi - \nu s\right) g(\theta, \varphi)$$

one can readily verify that one can choose the most natural measure and put

$$(f_1, f_2) = \int_0^{\pi}\int_0^{2\pi} \overline{g_1(\theta, \varphi)}\, g_2(\theta, \varphi) \sin\theta\, d\theta d\varphi, \tag{23}$$

where g_1 and g_2 are periodic in φ if m is even and anti-periodic in φ if m is odd.

For $m = 0$ and $\nu = i\alpha$ it is not possible to satisfy the relations (22) by the scalar product (23), since terms like $\alpha \cos\theta$, $\alpha \sin\theta \cos\varphi$ and $\alpha \sin\theta \sin\varphi$ in eqs. (11) are hermitian. Let us therefore try the most general bilinear form

$$(f_1, f_2) = \iiiint \overline{g_1(\theta, \varphi)}\, K(\theta, \theta', \varphi, \varphi',)\, g_2(\theta', \varphi') \sin\theta \sin\theta'\, d\theta d\theta' d\varphi d\varphi', \tag{24}$$

where

$$f_1 = e^{-\alpha s} g_1(\theta, \varphi),\ f_2 = e^{-\alpha s} g_2(\theta, \varphi),$$

and K is a function to be determined by the eqs. (22). From the first of these we find that K can depend only on

$$y = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varphi - \varphi').$$

From the second set of relations in eqs. (22) one finds that

$$K = C_\alpha (1 - y)^{\alpha - 1}. \tag{25}$$

Now the integral (24) converges only if α is positive, since $1-y$ goes as the distance between two points on the sphere squared. For the moment we therefore consider only such values of α, but later on we shall find that one can give a meaning to eq. (24) also for negative values of α. We must also make sure that

$$(f,f)\geqslant 0.$$

For this we expand in spherical functions and Legendre polynomials

$$f(\theta,\varphi)=\sum_{l,n} f_{ln}\, Y_l^n(\theta,\varphi), \tag{26}$$

$$(1-y)^{\alpha-1}=\sum_l \frac{2l+1}{2}\, b_l\, P_l(y). \tag{27}$$

$$\text{Thus}\quad (f,f)=C_\alpha \sum_{l'n'}\sum_{l''n''}\sum_l \frac{2l+1}{2}\, b_l\, f_{l'n'}\, \bar f_{l''n''}$$

$$\iiiint Y_{l'}^{n'}(\theta',\varphi')\, \bar Y_{l''}^{n''}(\theta'',\varphi'')\, P_l(y)\sin\theta'\sin\theta''\, d\theta'\, d\theta''\, d\varphi'\, d\varphi''. \tag{28}$$

The addition theorem for Legendre polynomials yields

$$P_l(y)=\sum_n \frac{4\pi}{2l+1}\, Y_l^n(\theta',\varphi')\, Y_l^{-n}(\theta'',\varphi'') \tag{29}$$

so that
$$(f,f)=C_\alpha\sum_{l,n} 2\pi b_l\, f_{ln}\, f_{l-n}=C_\alpha\sum_{l,n} 2\pi b_l\, |f_{ln}|^2 \tag{30}$$

since $\bar f_{ln}=f_{l-n}$. Thus
$$(f,f)\geqslant 0$$

if and only if
$$C_\alpha b_l\geqslant 0. \tag{31}$$

From eq. (27) we have

$$b_l=\int_{-1}^{1} \frac{P_l(y)}{(1-y)^{1-\alpha}}\, dy. \tag{32}$$

This integral can be calculated by using the generating function for $P_l(y)$. We get

$$\left.\begin{aligned} b_0&=\frac{2^\alpha}{\alpha},\\ b_l&=\frac{2^\alpha(1-\alpha)(2-\alpha)\dots(l-\alpha)}{\alpha(1+\alpha)(2+\alpha)\dots(l+\alpha)}.\end{aligned}\right\} \tag{33}$$

By choosing $C_\alpha=\alpha$ we see that eq. (31) can be satisfied for $|\alpha|\leqslant 1$. For $\alpha\to 0$, $C_\alpha b_l\to 1$, and we get the scalar product (23) as a limiting case. Negative values of α cause no trouble when we define the scalar product by eq. (30).

Obviously these values yield measure functions K which are more singular in angular space than a δ-function. Of course, our Hilbert space is complete in the norm. Thus not all functions g which belong to say the space characterized by $\alpha = 0$ are vectors in the space with negative α.

There remains one function space to be examined, namely the case (II c). The functions of this space are

$$f = sg(\theta, \varphi) + h(\theta, \varphi).$$

We make the following ansatz

$$(f_1, f_2) = \int_0^{\pi} \int_0^{2\pi} [a_{11} g_1 g_2 + a_{12} g_1 h_2 + a_{21} g_2 h_1 + a_{22} h_1 h_2] \sin \theta \, d\theta d\varphi. \tag{34}$$

Again using the relations (22) one finds that

$$a_{12} = -a_{21}, \quad a_{22} = 0.$$

From $(f, f) \geqslant 0$ it is obvious that $a_{11} \geqslant 0$. Thus the scalar product

$$(f_1, f_2) = \int_0^{\pi} \int_0^{2\pi} [a_{11} g_1 g_2 + a_{12} (g_1 h_2 - g_2 h_1)] \sin \theta \, d\theta d\varphi$$

leads to unitary representations. But all functions $f = h(\theta, \varphi)$ are equivalent to the zero vector and consequently the Hilbert space has only one "component" namely the one represented by $g(\theta, \varphi)$. The operator $\partial/\partial s$ again acts as the zero operator.

So far we have been able to define scalar products so that our representations are unitary in case that m is an arbitrary non-negative integer and ν real (the main series of unitary representations) and in case m is zero and $\nu = i\alpha$ where $0 < |\alpha| \leqslant 1$ (the supplementary series of unitary representations). Now Neumark [1] shows that one pair of the invariants r and q uniquely determine one irreducible representation up to unitary equivalence. Therefore we know that we have obtained all unitary representations up to unitary equivalence since the invariants can only take on the values [1]

$$r = 1 - k_0^2 + c^2,$$

$$q = k_0 c,$$

where k_0 is a non-negative integer or half integer and c is real or $k_0 = 0$ and $-1 \leqslant c^2 < 0$, and these values are also obtained by letting m and ν vary over their ranges. Furthermore one sees that for $m = 0$ the representations which differ in sign of ν or α are unitarily equivalent. This can also be shown by constructing the isometric transformation, which mediates the equivalence.

5. Conclusion

We have proposed to write the unitary irreducible representations of the homogeneous Lorentz group in an explicit form by means of a function space over the carrier space of restricted six-vectors. Since the rank of the Lie-algebra is two and we have two invariants, one has to use at least a four-dimensional carrier space if one wants to see explicitely how the irreducibility condition enters. If the carrier space can be identified with a coordinate space for a particle then one may hope to be able to give a meaning to the group invariants as well as to the integration in the scalar product. Especially the scalar product of non-local character for the supplementary series is then of interest.

ACKNOWLEDGEMENT

I wish to thank Professor N. Svartholm for his constant interest and valuable criticism. Dr. J. Nilsson has read the manuscript and contributed to a clearer formulation.

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Tryckt den 28 augusti 1964

Uppsala 1964. Almqvist & Wiksells Boktryckeri AB

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ARNE KIHLBERG

On the unitary representations of a class of pseudo-orthogonal groups

ALMQVIST & WIKSELL

STOCKHOLM

GÖTEBORG · UPPSALA

1965

ARKIV FÖR FYSIK Band 30 nr 11

3.65 08 3 Communicated 12 May 1965 by L. Hulthén and N. Svartholm

On the unitary representations of a class of pseudo-orthogonal groups

By Arne Kihlberg

ABSTRACT

A method for explicit construction of the unitary, irreducible representations of the pseudo-orthogonal groups $L(p,q)$ is presented. It is based on a realization of the elements of the Lie-algebra as differential operators on a carrier space. This space is the product space of the group spaces of the maximal compact subgroup K and an Abelian subgroup A both in $L(p,q)$. The Hilbert space is constructed as a function space on the carrier space. In an irreducible representation the vectors in the Hilbert space have a fixed dependence on the parameters of A and the scalar product is defined in terms of integration over the parameters of K. The method is particularly simple to apply when $q-p$ is 0, 1 or 2 since then the number of parameters of the carrier space is equal to the number of labels required to characterize a vector.

1. Introduction

The pseudo-orthogonal group in $p+q$ variables is defined as the group of real, linear, homogeneous transformations which leave invariant the quadratic form

$$x_1^2+x_2^2+\dots x_p^2-x_{p+1}^2-\dots-x_{p+q}^2. \tag{1}$$

It is a Lie-group of $\binom{p+q}{2}$ parameters. We may assume that both p and q are positive integers for if p or q is zero the group is an orthogonal group. Contrary to the orthogonal groups the pseudo-orthogonal ones are non-compact. This means among other things that the irreducible unitary representations are infinite dimensional. The pseudo-orthogonal groups can just as the homogeneous Lorentz group be divided into four components [1]. The first is continuously connected with the identity, while the other three include a reflection in the p first, or in the q last variables, or in both. In what follows we shall always deal only with the first, the identity component. For this subgroup we use the symbol $L(p,q)$.

In physical theories the pseudo-orthogonal groups have appeared in many different connections. Of course the most famous pseudo-orthogonal group is the homogeneous Lorentz group $L(1,3)$, which needs no further presentation. But also the groups $L(1,4)$ and $L(2,3)$ have been studied in quite a detail. These groups, usually called the de Sitter groups, are extensions of the homogeneous Lorentz group with a fifth space- or timelike dimension. They then allow for the introduction of a curvature constant as is suggested by general relativity. The pseudo-rotations in the planes (x_5, x_i) are connected with the translations in the four coordinates x_i. The de Sitter groups are thus to be regarded as generalizations of the Poincaré group. In a more indirect way the group $L(1,2)$ is also of physical interest. When determining the unitary irreducible representations of the Poincaré group one has according to Wigner [2], to consider certain subgroups, the little groups. An irreducible unitary representation of the little group and the mass value determine an irreducible unitary representation of the whole group. When the mass is imaginary the little group is $L(1,2)$. These representations with imaginary mass have however so far not been interpreted physically. Still another example of a pseudo-orthogonal group which has been suggested as a useful group at various times during the last fifty years is the conformal group. It is isomorphic to $L(2,4)$.

In the last few years the group theoretical treatment of elementary particle physics has been concentrated on the problem of finding internal symmetry groups. The interest has so far been mainly confined to compact groups, and this is presumably to a certain extent due to the absence of a simple representation theory for non-compact groups.

In the physical application of all the groups mentioned above one is interested in their unitary representations. (That the finite dimensional non-unitary representations of the homogeneous Lorentz group have played such a big role is due to the fact that the latter is a subgroup of the Poincaré group. Some unitary representations of this group can be given with the help of the finite dimensional representations of the homogeneous Lorentz group.) The unitary irreducible representations of the homogeneous Lorentz group were determined by Gelfand and Naimark [3] and Bargmann [4] in 1947 and those of $L(1,2)$ by Bargmann [4]. The group $L(1,4)$ was dealt with by Thomas [5] in 1941 and later on by Newton [6] and very thoroughly by Dixmier [7]. The de Sitter group $L(2,3)$ has been treated by Ehrman [8] in 1956. Esteve and Sona [9] have applied the theory of Graev [10] to the conformal group.

In all the solved examples above, except $L(2,3)$ and $L(2,4)$, one has used the technique to determine a representation by reducing it out with respect to irreducible representations of a compact subgroup. In the case $L(1,3)$ one uses the compact three-dimensional rotation group and for $L(1,4)$ the four-dimensional rotation group. Now it can easily be seen that this method has limitations. It is essential for the technique that each irreducible representation of the compact subgroup occurs only once. For this to be the case the subgroup has to be large enough. This can be seen as follows. In order to label the vectors in the representation space one needs a maximal set of commuting operators taken from the enveloping algebra of the Lie algebra of the group. A number of these can be chosen as the invariants. The pseudo-orthogonal group in $p+q$ variables has $(p+q-1)/2$ invariants if $p+q$ is odd and $(p+q)/2$ if $p+q$ is even. The number of parameters is $\binom{p+q}{2}$.

Therefore one needs [11]

$$\frac{1}{2}\left[\binom{p+q}{2}-\frac{p+q-1}{2}\right]$$

operators in the odd case and

$$\frac{1}{2}\left[\binom{p+q}{2}-\frac{p+q}{2}\right]$$

in the even case to label the vectors within an irreducible representation and

$$\frac{1}{2}\left[\binom{p+q}{2}+\frac{p+q-1}{2}\right] \quad (p+q \text{ odd}),$$

$$\frac{1}{2}\left[\binom{p+q}{2}+\frac{p+q}{2}\right] \quad (p+q \text{ even})$$

operators to label the vectors throughout any representation. Now the question is whether one can find a sufficient number of labels in the maximal compact subgroup. For the pseudo-orthogonal group the maximal compact subgroup is $SO_p \otimes SO_q$ where SO_p is the rotation group in p dimensions. We have to distinguish the four cases: $p+q$ odd or even, p odd or even.

We first consider the case $p+q$ odd, p odd, q even ($p<q$). Then to label the vectors within an irreducible representation of $L(p,q)$ with the operators of $SO_p \otimes SO_q$ we must have

$$\frac{1}{2}\left[\binom{p+q}{2}-\frac{p+q-1}{2}\right] \leqslant \frac{1}{2}\left[\binom{p}{2}+\frac{p-1}{2}\right]+\frac{1}{2}\left[\binom{q}{2}+\frac{q}{2}\right]$$

which gives

$$(p-1)\,(q-1) \leqslant 0.$$

This inequality can be satisfied only if $p=1$.

In the other cases one has similar restrictions and one finds that the reduction method may work when $p=1$ (or $q=1$) and in the case $p=q=2$.[1] Actually it has been proved that it always works in the case $p=1$ [12]. In all other cases one has to use other methods. The de Sitter group $L(2,3)$ does not fulfill these requirements so in this case one has to use a more general method. In fact Ehrman applies Harish-Chandra's general theory for arbitrary semi-simple groups.

In Harish-Chandra's [13, 14] theory one uses a Hilbert space whose vectors are functions on the maximal compact subgroup. This means that the vectors are functions of the parameters of that subgroup. One can see that this approach might be more successful since now the requirement that the labels within an irreducible representation should be less numerous than the number of parameters of $SO_p \otimes SO_q$ reads

$$\frac{1}{2}\left[\binom{p+q}{2}-\frac{p+q-1}{2}\right] \leqslant \binom{p}{2}+\binom{q}{2} \quad \text{for } p+q \text{ odd and}$$

$$\frac{1}{2}\left[\binom{p+q}{2}-\frac{p+q}{2}\right] \leqslant \binom{p}{2}+\binom{q}{2} \quad \text{for } p+q \text{ even.}$$

Both inequalities are always satisfied. On this Hilbert space Harish-Chandra defines a set of representations which need neither be unitary nor irreducible. He furthermore shows that every unitary irreducible representation is equivalent to one which can be obtained from the set. As can be seen from the paper by Ehrman the last step, namely the selection of those representations which are irreducible and unitary, is far from simple. Also the Harish-Chandra theory is rather abstract and physicists would perhaps like to have a more explicit way of constructing the representations.

It is our purpose to give such a method of constructing the unitary representations of $L(p,q)$. It will turn out that it is closely connected to the theory of Harish-Chandra.

[1] $L(2,2)$ is locally isomorphic to $L(1,2) \otimes L(1,2)$; see appendix.

We use the maximal compact subgroup as a carrier space for the Hilbert space and the invariants are expressed in terms of first order operators which in their turn are connected to a second subgroup. However we deviate from Harish-Chandra in the the following respects. The scalar product in the Hilbert space is not given beforehand but is rather adjusted so that the representation becomes unitary. Thus one of the problems mentioned above, namely the selection of those representations which are equivalent to unitary representations is absent in our theory. Furthermore our approach is based on rather simple considerations of transformations in the group space which therefore makes it much more limited in application than the general theory of Harish-Chandra. Nevertheless we think that our treatment might be of interest in a number of applications simply because the mathematics involved is of a much less elaborate kind.

In section 2 we derive some properties of the Lie-algebra especially with emphasis on its enveloping algebra and the invariants. In section 3 we introduce a canonical division of any semi-simple Lie-group into three subgroups. The property of this division makes it possible to map the group elements into certain transformations in the group space modulo the last subgroup. The group space of the first two subgroups will serve as carrier space for the representation space. Thereby the second subgroup will be used to express the invariants. In section 4 we introduce the scalar product into the representation space which makes it possible to classify the unitary representations. Finally in the appendix we determine, as an example, the unitary representations of $L(2,2)$.

2. The Lie-algebra and its enveloping algebra

The Lie-algebra of the pseudo-orthogonal group $L(p,q)$ can easily be found by considering the one-parameter subgroups of rotations or pseudo-rotations in all coordinate planes (x_i, x_j). For instance the subgroup of pseudo-rotations in the $(1, p+1)$-plane consists of the matrices

$$g(u) = \left|\begin{array}{cccc:cccc} \cosh u & & & & \sinh u & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hdashline \sinh u & & & & \cosh u & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right| \qquad -\infty < u < \infty .$$

The corresponding element in the Lie-algebra is formed by taking the limit

$$\lim_{u\to 0} \frac{g(u)-1}{u} = \left|\begin{array}{cc:cc} & & 1 & \\ & & & \\ \hdashline 1 & & & \\ & & & \end{array}\right| .$$

The whole Lie-algebra is generated by all matrices

$$\left|\begin{array}{cccc:ccc} 0 & a_{12} & a_{13} & \cdots & b_{11} & \cdots & b_{1q} \\ -a_{12} & 0 & & & \vdots & & \vdots \\ -a_{13} & & \ddots & & & & \\ \vdots & & & 0 & b_{q1} & \cdots & b_{qq} \\ \hdashline b_{11} & \cdots & & b_{q1} & 0 & -f_{12} & \cdots \\ \vdots & & & \vdots & f_{12} & 0 & \\ & & & & \vdots & & \ddots \\ b_{1q} & \cdots & & b_{qq} & & & 0 \end{array}\right|,$$

where the numbers a_{ij}, b_{ij}, f_{ij} are real. We can obviously choose a basis L_{ij} in the following way

$$\begin{aligned} L_{ij} &= e_{ij} - e_{ji} \quad \text{for} \quad i, j \leqslant p \\ L_{ij} &= -e_{ij} + e_{ji} \quad \text{for} \quad i, j > p \\ L_{ij} &= e_{ij} + e_{ji} \quad \text{for} \quad i \leqslant p, j > p \\ L_{ij} &= -e_{ij} - e_{ji} \quad \text{for} \quad i > p, j \leqslant p, \end{aligned}$$

where the matrix e_{ij} has a 1 at the position (ij) and zeros elsewhere. By introducing the metric tensor

$$\gamma_{ij} = \left|\begin{array}{cccc:cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hdashline & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & \ddots & \\ & & & & & & & -1 \end{array}\right|,$$

the commutation relations of the Lie-algebra of the pseudo-orthogonal group can be written

$$[L_{ij}, L_{kl}] = -\gamma_{ik} L_{jl} - \gamma_{jl} L_{ik} + \gamma_{il} L_{jk} + \gamma_{jk} L_{il}. \tag{2}$$

The generators L_{ij} for $i, j \leqslant p$ generate rotations in the first p variables. They form a subalgebra which is isomorphic to the Lie-algebra of the rotation group in p variables. Similarly L_{ij}, $i, j > p$ form the Lie-algebra of rotations in q variables. These two subalgebras together generate the maximal compact subgroup of $L(p,q)$ which is the direct product $SO_p \otimes SO_q$ of the two rotation groups. The generators L_{ij} for $i \leqslant p$, $j > p$ correspond to accelerations in one "time-like" and one "space-like" variable. These generators are the "imaginary" counterparts of the corresponding operators of SO_{p+q} the compact group belonging to the same complex group as $L(p,q)$.

An important tool for the classification of the irreducible representations of a Lie-group or a Lie-algebra is the universal enveloping algebra, see e.g. [15]. Its elements are the equivalence classes of polynomials of the generators of the Lie-algebra.

Two polynomials are said to be equivalent if they can be transformed into each other with the help of the commutation relations. The centre of the universal enveloping algebra consists of those elements which commute with all other elements. This means that it commutes with the generators of the Lie-algebra. The elements of the centre are spanned by the invariants of the Lie-algebra. For the rotation group SO_n the invariants are [16] (F_{ij}: the generators fulfilling eq. 2 with $\gamma_{ij}=\delta_{ij}$)

$$\left.\begin{aligned} I^2 &= \sum_i \sum_j F_{ij} F_{ji} \\ I^4 &= \sum_i \sum_j \sum_k \sum_l F_{ij} F_{jk} F_{kl} F_{li}, \ \ldots \\ I^k &= \sum_{i_1} \ldots \sum_{i_k} F_{i_1 i_2} F_{i_2 i_3} \ldots F_{i_{k-1} i_k} F_{i_k i_1}, \end{aligned}\right\} \tag{3}$$

where $k=n-1$ if n is odd and $k=n-2$ if n is even. In the case of even n there is one more invariant

$$I^{\frac{1}{2}} = \sum_{i_1} \ldots \sum_{i_n} \varepsilon^{i_1 i_2 \ldots i_n} F_{i_1 i_2} F_{i_3 i_4} \ldots F_{i_{n-1} i_n}, \tag{4}$$

where $\varepsilon^{ij}\ldots$ is the totally antisymmetric tensor in the indices $1\ldots n$. The invariants of $L(p,q)$ are obtained from those of SO_{p+q} by replacing F_{ij} by iL_{ij} if $i \leqslant p, j > p$ or $i > p, j \leqslant p$ and F_{ij} by L_{ij} in other cases.

The number of terms in I^j grows very fast with j. Instead of I^j one can use another invariant Δ^j which involves fewer terms. Δ^j is defined as the sum of all principal minors of the order j of the $n \times n$ matrix $\{F_{ij}\}$. (A principal minor of order j is obtained as the determinant of the matrix which remains when one removes the corresponding $n-j$ rows and columns.)

In an irreducible representation of the group $L(p,q)$ the invariants must have constant values. They therefore help to distinguish the different inequivalent irreducible representations although there may be inequivalent representations which have the same values for the invariants [3, 4, 8].

3. Realization of a semi-simple group by means of transformations in the group space. The carrier space for the representation

As pointed out in the introduction $L(p,q)$ is the identity component of the pseudo-orthogonal group. Further insight into the topological properties of $L(p,q)$ can be obtained from a lemma by Iwasawa [17] concerning arbitrary semi-simple Lie-groups.

Lemma 1. Let G^* be the adjoint group of a real semi-simple Lie-group. Then there exists a connected and simply connected solvable subgroup H and a maximal compact subgroup K^* of G^* such that

$$G^* = K^* H = HK^*, \ H \cap K^* = \{e\},$$

i.e., any element g^* of G^* can be written uniquely in the form

$$g^* = k^* h = h' k^{*\prime}, \ h, h' \in H, \ k^*, k^{*\prime} \in K^*$$

and h, h' and k^*, $k^{*\prime}$ depend continuously on g^*. Also the space of G^* is the Cartesian product of the spaces of H and K^* of which the former is homeomorphic to an Euclidean space.

Now the adjoint group G^* coincides with the original group G if this latter does not have any (discrete) centre. This is the case for $L(p,q)$ unless both p and q are even. It is easy to prove that an element of $L(p,q)$ which is in the centre has to be a multiple of unity. Thus the centre can at most contain two elements ± 1. However if p or q is odd the element -1 is a reflection in the p first or q last variables. Thus it does not belong to the connected component. For these groups the lemma is immediately applicable and we therefore have the useful information that the connectivity is given by that of the maximal compact subgroup which is $SO_p \otimes SO_q$. When p and q are even we have a centre of order two. This is however contained in the inverse image K of K^* in the mapping $G \to G^*$ [17]. Thus the conclusions of the lemma are still true if one replaces G^* by G and K^* by K. Now the group SO_p is doubly connected if $p > 2$ while for $p = 2$ it is infinitely connected. As has been remarked by Wigner [18] the de Sitter group $L(2,3)$ therefore differs radically from the group $L(1,4)$ in topology and this has the consequence of introducing an extra invariant in the representations up to a factor.

The division in lemma 1 can be carried a step further as has been shown by Harish-Chandra [14].

Lemma 2. Let G be a connected semi-simple Lie group with the Lie-algebra $\mathbf{g}$. Then $\mathbf{g}$ can be written

$$\mathbf{g} = \mathbf{k} + \mathbf{h}^+ + \mathbf{n},$$

where $\mathbf{h}^+$ is a maximal Abelian subalgebra generated by such elements which become "imaginary" when passing from the compact semi-simple Lie-algebra to the chosen non-compact one. $\mathbf{h}^+$ forms part of a Cartan subalgebra. $\mathbf{k}$ is the Lie-algebra of K^* in lemma 1 and the subalgebra $\mathbf{n}$ is generated from a set of operators which are associated with certain positive roots in the Cartan root diagram. Furthermore let A and N be those subgroups of G which correspond to $\mathbf{h}^+$ and $\mathbf{n}$. $AN = H$ is then the subgroup of lemma 1 and N is an invariant subgroup of H. If furthermore K is the inverse image of K^*, the maximal compact subgroup of G^*, in the mapping $G \to G^*$, then every element g of G can be written uniquely in the form

$$g = k\ a\ n, \text{ where } k \in K,\ a \in A,\ n \in N.$$

When applying the result of lemma 2 to the pseudo-orthogonal groups one actually does not need the whole theory of Cartan's classification of semi-simple groups. We have already noted that $K = SO_p \otimes SO_q$. For A one has just to choose a maximal Abelian group of accelerations i.e. transformations involving one timelike and one spacelike variable. Then one quite easily finds out what are the generators of N. These have to be formed as linear combinations of the accelerations which are not in A and of generators of K. One has to apply the restriction that N is an invariant subgroup of H.

This division of $L(p,q)$ into three subgroups of which the first is maximal compact and the last two together form a noncompact subgroup will now be the basis for a certain parametrization of $L(p,q)$ and will also make it possible to obtain very convenient expressions for the operators of its Lie-algebra. It is well known that an arbitrary Lie-group can be represented as a transformation group on its parameters. This can be accomplished by left multiplication

$$g \xrightarrow{g_1} g_1^{-1} g, \quad \text{all} \quad g \in G,$$

which associates with every element $g_1 \in G$ a transformation of the point g in the group space. By considering infinitesimal elements g_1 one can represent the elements of the corresponding Lie-algebra as differential operators in the group parameters. In general this procedure leads to very cumbersome expressions and as will be shown below it is also not necessary to use all parameters of the group. Let us see what happens to the three subgroups under left multiplication by an element which belongs to K. We get $k'(k_1 a_1 n_1) = (k' k_1) a_1 n_1$ if $k' \in K$, by the uniqueness of the division. Therefore we have a change only in the parameters of K. All generators of the Lie-algebra of K are therefore differential operators in the parameters $(\varphi, \theta \ldots)$ of K. Denoting a generator by capital L we have

$$L_K = f_K^{\varphi}(\varphi, \theta \ldots)\frac{\partial}{\partial \varphi} + f_K^{\theta}(\varphi, \theta \ldots)\frac{\partial}{\partial \theta} + \ldots. \tag{5}$$

Let us next consider multiplication by an element of A. We get

$$a'(k_1 a_1 n_1) = k_{k_1 a'} a_{k_1 a'} n_{k_1 a'} a_1 n_1 = k_{k_1 a'} (a_{k_1 a'} a_1)(n_{k_1 a' a_1} n_1),$$

where $k_{k_1 a'}$ is a function of k_1 and a'. Here we have used the fact that A is a factor-group of AN. Therefore we get a transformed k_1, a_1 and n_1. But the transformations of k_1 and a_1 depend only on k_1 not on a_1 or n_1 as is seen from the formula. Note that A is Abelian. This shows that the infinitesimal generators of A can be written

$$L_A = f_A^{\varphi}(\varphi, \theta \ldots)\frac{\partial}{\partial \varphi} + \ldots f_A^{\lambda}(\varphi, \theta \ldots)\frac{\partial}{\partial \lambda} + \ldots + f_A^{r}(\varphi, \theta \ldots \lambda, \mu \ldots r, s \ldots)\frac{\partial}{\partial r} + \ldots \tag{6}$$

if the parameters of A are denoted $\lambda, \mu \ldots$ and those of N by $r, s \ldots$. Finally let us consider a left multiplication by an element from N,

$$n'(k_1 a_1 n_1) = k_{k_1 n'} a_{k_1 n'} n_{k_1 n'} a_1 n_1 = k_{k_1 n'} (a_{k_1 n'} a_1)(n_{k_1 n' a_1} n_1).$$

This formula shows similarly that

$$L_N = f_N^{\varphi}(\varphi, \theta \ldots)\frac{\partial}{\partial \varphi} + \ldots f_N^{\lambda}(\varphi, \theta \ldots)\frac{\partial}{\partial \lambda} + \ldots + f_N^{r}(\varphi, \theta \ldots \lambda, \mu \ldots r, s \ldots)\frac{\partial}{\partial r} + \ldots. \tag{7}$$

Thus we find that for all the generators of the Lie-algebra only the derivatives in the parameters of N have coefficients depending on all group parameters. This means that we can omit all these terms and still have the same commutation relations. Expressed in terms of the groups one can say that we realize the group as a group of transformations in the group space modulo the subgroup N. The product space of the group spaces of K and A shall be termed the carrier space of the Lie-algebra since the operators of it are expressed as linear differential operators on this space. Of course the possibility of considering this reduced carrier space instead of the whole group space of G considerably simplifies the expressions for the generators. The question is therefore if it perhaps simplifies them too much so that the realization makes it impossible to obtain all representations. This is what happens when one puts

$$M_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} \tag{$^+$}$$

for the generators of the homogeneous Lorentz group. Because of the invariant equation

$$x^\mu x_\mu = \text{Const.}$$

this is a realization of the generators $M_{\mu\nu}$ by three parameters. For the homogeneous Lorentz group one needs a four-dimensional carrier space [19] and therefore it is not surprising that one does not obtain all representations of $L(1,3)$ from the parametrization ($^+$). In the introduction we found that one needs a carrier space of dimension

$$\begin{aligned} &\tfrac{1}{4}[(p+q)^2-1] \quad &&\text{if} \quad p+q \text{ is odd,} \\ &\tfrac{1}{4}(p+q)^2 &&\text{if} \quad p+q \text{ is even.} \end{aligned}$$

Now the dimension of A is p if $p \leqslant q$ and therefore the dimension of N is $p(q-1)$ since in the pseudo-orthogonal group there is pq "imaginary" operators, the accelerations, as compared to the corresponding compact group SO_{p+q}. The requirement that the carrier space has sufficiently many dimensions therefore is

$$\begin{aligned} &\tfrac{1}{4}[(p+q)^2-1] \leqslant \binom{p+q}{2} - p(q-1) \qquad &&, p+q \text{ odd}, \ p<q, \\ &\tfrac{1}{4}(p+q)^2 \leqslant \binom{p+q}{2} - p(q-1) &&, p+q \text{ even}, \ p \leqslant q. \end{aligned}$$

Both inequalities are always fulfilled. We therefore conclude that provided there are no invariant equations in the remaining parameters one can always take the group space modulo the subgroup N as carrier space for the transformations of the group and the Lie-algebra.

The case when one has equality in the above equations is of special interest and we will mainly be concerned with the corresponding groups in the rest of the paper. The condition for equality is

$$\begin{aligned} &p = q-1 \quad \text{if } p+q \text{ is odd and } p<q \qquad \text{and} \\ &\left.\begin{matrix} p=q \\ p=q-2 \end{matrix}\right. \quad \text{if } p+q \text{ is even and } p \leqslant q. \end{aligned} \tag{8}$$

The subgroup A also plays a special role in the decomposition of G since its parameters do not enter into the coefficients of the derivatives in the reduced generators. Therefore the values of $\partial/\partial\lambda$, $\partial/\partial\mu \dots$ can be put equal to constants $a, b \dots$. These constants will then enter into the invariants of the Lie-algebra. Let us see when the number of invariants and the number of parameters of A coincide. For this to be the case we must have

$$\frac{p+q-1}{2}=p \quad \text{if } p+q \text{ is odd,}$$

$$\frac{p+q}{2}=p \qquad \text{if } p+q \text{ is even.}$$

We thus find that for $p=q-1$ and $p=q$, the carrier space has the right dimension and the subgroup A has the dimension equal to the number of invariants of the group. In this special case the invariants of the Lie-algebra can all be expressed as polynomials in the derivatives $\partial/\partial\lambda$, $\partial/\partial\mu$..... What about the case $p=q-2$ when $p+q$ is even? In this case there is one invariant more than there are dimensions in A. But in this case the subgroup K has one dimension too much since we want to use its parameters in connection with the labelling of the vectors inside an irreducible representation. The dimension of K is p^2+p+1 if $p=q-2$ while the required number to label the vectors is p^2+p. Therefore it should in this case be possible to "move one parameter from K to A". In fact, as was stated in lemma 2, A is a part of a Cartan subalgebra the rest of which is to be found in K. Now since the number of invariants is equal to the dimension of a Cartan subalgebra it must be possible to find in K the required number of one parameter subgroups which commute with A. By means of a suitable parametrization of K one can achieve that the parameters of these also do not enter into the coefficients of the derivatives.

In all the cases (8) we have found that the Lie-algebra can be given with the help of differential operators in which the invariants enter merely as parameters a, b.... The invariants themselves are polynomials in a, b.... The special case $p=q-2$ differs from the others in that one of the constants a, b... takes only discrete values. This can be seen from the fact that the corresponding differential operator $\partial/\partial\psi$ is an operator in a compact variable ψ of K.

Even in the cases not covered by (8) the parametrization we have described may be useful. One then however has more parameters than necessary, but this difficulty might be possible to handle with a suitable constraint.

4. Unitary representations

In physical applications one is as a rule interested in unitary representations up to a factor. This means that the unitary operators $U(g)$ satisfy

$$U(g_1)\ U(g_2)=\omega(g_1,g_2)\ U(g_1\, g_2),$$

where ω is a function of g_1 and g_2 such that $|\omega|=1$. Now Bargmann [20] has shown that for the pseudo-orthogonal groups $L(p,q)$ this means that one has to look for the unitary representations of the universal covering group of $L(p,q)$. The universal covering group $\bar{L}(p,q)$ is simply connected. Therefore in the case $p>2$ and $q>2$ we see from the lemma 1 that $\bar{L}(p,q)$ contains four sheets of $L(p,q)$ while for $p<2$ and $q>2$ it contains two. If $p=2$ or $q=2$, $\bar{L}(p,q)$ contains the space of $L(p,q)$ infinitely many times. The fact that one should study the representations of $\bar{L}(p,q)$ means that one has extra invariants for the representations. For the homogeneous Lorentz group this is the integer or half integer character of the spin-values. For the de Sitter group $L(2,3)$ this extra invariant has a continuous variation [8, 18].

From the works of Gårding, Harish-Chandra [13] and Dixmier [7] it is known that the search for unitary irreducible representations of $L(p,q)$ or $\bar{L}(p,q)$ can be reduced to the problem of finding the Hermitean, irreducible representations of the Lie-algebra.

Now we are going to discuss how one may obtain all unitary irreducible representations of the Lie-algebra of $L(p,q)$ with the help of the parametrization of the carrier space which we introduced in the foregoing section. We recall that we obtained the following general form of the generators of the Lie-algebra

$$L_{ij} = f_{ij}^{\varphi}(\varphi, \theta \ldots) \frac{\partial}{\partial \varphi} + f_{ij}^{\theta}(\varphi, \theta \ldots) \frac{\partial}{\partial \theta} + \ldots$$

if L_{ij} is a generator of K and

$$L_{ij} = f_{ij}^{\varphi}(\varphi, \theta \ldots) \frac{\partial}{\partial \varphi} + f_{ij}^{\theta}(\varphi, \theta \ldots) \frac{\partial}{\partial \theta} + \ldots + f_{ij}^{\lambda}(\varphi, \theta \ldots) \frac{\partial}{\partial \lambda} + \ldots$$

in the other cases. The parameters $\varphi, \theta \ldots$ belong to the subgroup K and $\lambda, \mu \ldots$ are the parameters of the subgroup A. The functions $f_{ij}^{\varphi} \ldots f_{ij}^{\lambda} \ldots$ depend only on the parameters of K. (In the case $p = q - 2$ they do not depend on the last of these parameters.)

As representation space we now choose a linear space of functions

$$\tilde{f}(\varphi, \theta \ldots \lambda, \mu \ldots).$$

By imposing that the space should be irreducible it is necessary (but not sufficient) that the functions $\tilde{f}$ are eigenfunctions of the invariants. But the invariants are polynomials of the derivatives $\partial/\partial\lambda, \partial/\partial\mu \ldots$ and therefore the functions in an irreducible representation space can be chosen as eigenfunctions of $\partial/\partial\lambda, \partial/\partial\mu \ldots$ i.e. they have the form

$$\tilde{f}(\varphi, \theta \ldots \lambda, \mu \ldots) = e^{ia\lambda} e^{ib\mu} \ldots f(\varphi, \theta \ldots),$$

where f now is a function only on K and $a, b \ldots$ are arbitrary complex numbers. The irreducibility condition thus fixes the dependence on the parameters λ, μ. Those which are left $\varphi, \theta \ldots$ will span the carrier space for the Hilbert space. By this we mean that we are going to define a scalar product involving integration over the variables $k = (\varphi, \theta \ldots)$. The most general scalar product involves a positive definite Hermitean kernal $M(k_1, k_2)$

$$(f_1, f_2) = \iint \overline{f_1(k_1)}\, M(k_1, k_2)\, f_2(k_2)\, dk_1\, dk_2,$$

where dk is the invariant volume element of K. The condition

$$(f_1, f_2) = \overline{(f_2, f_1)}$$

implies
$$M(k_1, k_2) = \overline{M(k_2, k_1)},$$

while the condition $(f, f) \geq 0$ for all f implies that M is positive definite. Having defined a scalar product the next step is to define the Hilbert space H as that linear set of functions $f(k)$ which have finite norm. The generators L_{ij} of the Lie-algebra are linear operators in H. We now require that iL_{ij} is Hermitean or L_{ij} anti-Hermitean. Since the L_{ij} are given this means that the kernal $M(k_1, k_2)$ has to be chosen in a certain form. It is easy to see what restrictions result from the requirement that the generators of K are anti-Hermitean. If L_{ij} is anti-Hermitean $U = e^{\varepsilon_{ij} L_{ij}}$ is unitary. However from the construction of L_{ij} this means that

$$f(k) \xrightarrow{k'} f'(k) = f(k'k)$$

has to be a unitary transformation for every $k' \in K$. Thus

$$\int\int \overline{f_1(k'k_1)}\, M(k_1, k_2)\, f_2(k'k_2)\, dk_1\, dk_2 = \int\int \overline{f_1(k_1)}\, M(k_1, k_2)\, f_2(k_2)\, dk_1\, dk_2,$$

which gives after change of variables and using that dk is an invariant measure on K

$$M(k_1 k'^{-1}, k_2 k'^{-1}) = M(k_1, k_2).$$

Therefore the kernel $M(k_1, k_2)$ depends only on $k_1 k_2^{-1}$. Consequently M has to be an invariant two-point measure. Of course the ordinary invariant one-point measure is contained as a special case when M is a δ-function in all its variables. However the two-point measure should be more natural in connection with bilinear constructions such as Hilbert spaces. It is therefore remarkable that they have not been used to a larger extent.

The remaining anti-Hermitean generators, which are all accelerations, put further conditions on M. These conditions are also dependent on the parameters $a, b \ldots$ which determine the invariants. For some values of $a, b \ldots$ it may be possible to find a positive definite kernel M, for others not. In this way one finds the possible ranges for the parameters $a, b \ldots$. Of course one has to admit kernels which are not functions but only distributions.

It is a general feature for real semi-simple groups that the irreducible representations can be grouped into several series [21]. Now it turns out that in one series, the main series, the kernel $M(k_1, k_2)$ reduces to a one-point measure, the ordinary invariant measure on K, while in the supplementary series the kernel M will depend on $a, b \ldots$.

We now give a short summary. A set of representations $D(a, b \ldots)$ parametrized by the complex parameters $a, b \ldots$ has been given. From the construction of the measure function M we know that $D(a, b \ldots)$ is unitary. However it may still be reducible. We can now compare this set of representations with the one defined by Harish-Chandra. In fact in his set a representation is parametrized by a linear function on a certain Cartan subalgebra which then involves parameters corresponding to $a, b \ldots$ Harish-Chandra chooses the measure M to be a one-point measure and instead he only requires the representation to be equivalent to a unitary representation. So in fact one can identify our set with his and then one also knows from his result that the procedure we have used is perfectly general: every irreducible unitary representation can be obtained from some $D(a, b \ldots)$ as an irreducible part.

APPENDIX

In this appendix we want to illustrate our method by constructing the unitary representations of a pseudo-orthogonal group. The group we choose, $L(2,2)$, is somewhat trivial since its Lie-algebra is the direct sum of the Lie-algebra of $L(1,2)$ with itself. Anyhow among the groups with $p+q=4$ there is only one more non-compact one, namely $L(1,3)$. This group has been treated with our method in an earlier paper [19], although the realization of the Lie-algebra was obtained from a physical starting point. The groups with $p+q=5$ at once become quite involved. Therefore there remains only $L(1,3)$ and $L(2,2)$ which are simple enough to serve as illustrations.

The group $L(2,2)$ leaves invariant the quadratic form

$$x_1^2+x_2^2-x_3^2-x_4^2.$$

It consists of all real 4×4 matrices B, which are continuously connected to the identity, and which fulfill

$$B^T\gamma B=\gamma,$$

where γ is the matrix

$$\gamma=\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{vmatrix}.$$

The group has a centre of order two, which consists of the matrices $\pm I$. The basis elements L_{ij} of the Lie-algebra are given as in section 2. L_{12} and L_{34} generate the maximal compact subgroup $K=SO_2\otimes SO_2$. In order to find the generators of A, one has to look for a maximal Abelian subalgebra among the accelerations. This algebra is two-dimensional, and we choose as its generating elements L_{23} and L_{14}. The subgroup N finally must also be two-dimensional, and it is easy to find that a possible choice for its generators is $N_1=L_{13}-L_{12}$ and $N_2=L_{24}-L_{34}$.

The general group element g of $L(2,2)$ can now be parametrized as follows

$$g=e^{\varphi L_{12}}\,e^{\psi L_{34}}\,e^{\mu L_{23}}\,e^{\nu L_{14}}\,e^{sN_1}\,e^{tN_2},$$

where
$$0\leqslant\varphi,\ \psi\leqslant 2\pi,\quad -\infty<\mu,\nu,s,t<\infty.$$

We have deliberately chosen a parametrization which is not adapted to the decomposition into $L^{(1)}(1,2)\otimes L^{(2)}(1,2)$ since this property is special for $L(2,2)$.

To shorten the expressions in the following we write $s\varphi$ for $\sin\varphi$, $c\varphi$ for $\cos\varphi$, $\mathrm{Sh}\mu$ for $\sinh\mu$ and $\mathrm{Ch}\mu$ for $\cosh\mu$.

The group element n of N has the form

$$n=e^{sN_1}\,e^{tN_2}=\left|\begin{array}{cc:cc} 1 & -s & s & 0 \\ s & 1-\dfrac{s^2}{2}+\dfrac{t^2}{2} & \dfrac{s^2}{2}-\dfrac{t^2}{2} & t \\ \hdashline s & -\dfrac{s^2}{2}+\dfrac{t^2}{2} & 1+\dfrac{s^2}{2}-\dfrac{t^2}{2} & t \\ 0 & t & -t & 1 \end{array}\right|,$$

and is completely analoguous to the Lorentz transformations of second kind studied by Wigner [2]. The general matrix g is

$$g = \left|\begin{array}{cc|cc}
\operatorname{Ch}\nu\, c\varphi + s\, e^{\mu} s\varphi & [-s \operatorname{Ch}\nu + t \operatorname{Sh}\nu]\, c\varphi + \left[\operatorname{Ch}\mu - e^{\mu}\left(\frac{s^2}{2} - \frac{t^2}{2}\right)\right] s\varphi & [s \operatorname{Ch}\nu - t \operatorname{Sh}\nu]\, c\varphi + \left[\operatorname{Sh}\mu + e^{\mu}\left(\frac{s^2}{2} - \frac{t^2}{2}\right)\right] s\varphi & \operatorname{Sh}\nu\, c\varphi + t\, e^{\mu} s\varphi \\
-\operatorname{Ch}\nu\, s\varphi + s\, e^{\mu} c\varphi & [s \operatorname{Ch}\nu - t \operatorname{Sh}\nu]\, s\varphi + \left[\operatorname{Ch}\mu - e^{\mu}\left(\frac{s^2}{2} - \frac{t^2}{2}\right)\right] c\varphi & [-s \operatorname{Ch}\nu + t \operatorname{Sh}\nu]\, s\varphi + \left[\operatorname{Sh}\mu + e^{\mu}\left(\frac{s^2}{2} - \frac{t^2}{2}\right)\right] c\varphi & -\operatorname{Sh}\nu\, s\varphi + t\, e^{\mu} c\varphi \\
\hline
-\operatorname{Sh}\nu\, s\psi + s\, e^{\mu} c\psi & [s \operatorname{Sh}\nu - t \operatorname{Ch}\nu]\, s\psi + \left[\operatorname{Sh}\mu - e^{\mu}\left(\frac{s^2}{2} - \frac{t^2}{2}\right)\right] c\psi & [-s \operatorname{Sh}\nu + t \operatorname{Ch}\nu]\, s\psi + \left[\operatorname{Ch}\mu + e^{\mu}\left(\frac{s^2}{2} - \frac{t^2}{2}\right)\right] c\psi & -\operatorname{Ch}\nu\, s\psi + t\, e^{\mu} c\psi \\
\operatorname{Sh}\nu\, c\psi + s\, e^{\mu} s\psi & [-s \operatorname{Sh}\nu + t \operatorname{Ch}\nu]\, c\psi + \left[\operatorname{Sh}\mu - e^{\mu}\left(\frac{s^2}{2} - \frac{t^2}{2}\right)\right] s\psi & [s \operatorname{Sh}\nu - t \operatorname{Ch}\nu]\, c\psi + \left[\operatorname{Ch}\mu + e^{\mu}\left(\frac{s^2}{2} - \frac{t^2}{2}\right)\right] s\psi & \operatorname{Ch}\nu\, c\psi + t\, e^{\mu} s\psi
\end{array}\right| .$$

According to the results of section 3 we now have to multiply g by the inverse g_1^{-1} of an element g_1. Then all parameters φ, ψ, μ, ν, s and t get changed to new values φ', ψ', μ', ν', s', t' in a new partition of the product $g_1^{-1}g = g'$. But the essential result of section 3 was that the change in φ, ψ, μ, ν depends only on φ and ψ, the parameters of the maximal compact subgroup K. Now let g_1 be infinitesimal and generated, say, by L_{23}. Thus

$$g_1^{-1} = e^{-\varepsilon L_{23}} = \left|\begin{array}{cc|cc}
1 & & & \\
 & 1 & -\varepsilon & \\
\hline
 & -\varepsilon & 1 & \\
 & & & 1
\end{array}\right| ; \quad |\varepsilon| \ll 1,$$

and we get 16 equations from $g_1^{-1}g = g'$ which determine $\varphi' \ldots t'$ in terms of $\varphi \ldots t$. For instance from the first column of g' we get

$$\begin{aligned}
&\operatorname{Ch}\nu\, c\varphi + s\, e^{\mu} s\varphi = \operatorname{Ch}\nu'\, c\varphi' + s'\, e^{\mu'} s\varphi' \\
&-\operatorname{Ch}\nu\, s\varphi + s\, e^{\mu} c\varphi - \varepsilon[-\operatorname{Sh}\nu\, s\psi + s\, e^{\mu} c\psi] = -\operatorname{Ch}\nu'\, s\varphi' + s'\, e^{\mu'} c\varphi' \\
&-\operatorname{Sh}\nu\, s\psi + s\, e^{\mu} c\psi - \varepsilon[-\operatorname{Ch}\nu\, s\varphi + s\, e^{\mu} c\varphi] = -\operatorname{Sh}\nu'\, s\psi' + s'\, e^{\mu'} c\psi' \\
&\operatorname{Sh}\nu\, c\psi + s\, e^{\mu} s\psi = \operatorname{Sh}\nu'\, c\psi' + s'\, e^{\mu'} s\psi'.
\end{aligned}$$

By dividing these equations by ε one gets four equations for the derivatives $\left.\frac{d\varphi}{d\varepsilon}\right|_{\varepsilon=0} \cdots \left.\frac{dt}{d\varepsilon}\right|_{\varepsilon=0}$. These can be solved to give

$$\left.\frac{d\nu}{d\varepsilon}\right|_{\varepsilon=0} = -s\varphi\, s\psi$$

$$\left.\frac{d\varphi}{d\varepsilon}\right|_{\varepsilon=0} = s\varphi\, c\psi$$

$$\left.\frac{d\psi}{d\varepsilon}\right|_{\varepsilon=0} = c\varphi\, s\psi,$$

if one also uses the fact that these derivatives should be independent of μ, ν, s and t. To determine $d\mu/d\varepsilon$ one has to use the other columns of g' as well. One finds

$$\left.\frac{d\mu}{d\varepsilon}\right|_{\varepsilon=0} = -c\varphi\, c\psi.$$

Therefore we have the following realization of the generator L_{23} as a differential operator on the group spaces of K and A

$$L_{23} = s\varphi\, c\psi \frac{\partial}{\partial\varphi} + c\varphi\, s\psi \frac{\partial}{\partial\psi} - c\varphi\, c\psi \frac{\partial}{\partial\mu} - s\varphi\, s\psi \frac{\partial}{\partial\nu}.$$

The realizations of L_{12} and L_{34} is much simpler since K is Abelian. We get

$$L_{12} = -\frac{\partial}{\partial\varphi}$$

$$L_{34} = -\frac{\partial}{\partial\psi}.$$

Of course one can continue to determine the other generators of the Lie-algebra in the same manner but it is actually simpler to derive the expressions for them by taking commutators between L_{23} and L_{12}, L_{34}. In this way one obtains the following realization of the Lie-algebra of $L(2,2)$ on the carrier space of K and A

$$\left.\begin{aligned}
L_{12} &= -\frac{\partial}{\partial\varphi} \\
L_{34} &= -\frac{\partial}{\partial\psi} \\
L_{23} &= s\varphi\, c\psi \frac{\partial}{\partial\varphi} + c\varphi\, s\psi \frac{\partial}{\partial\psi} - c\varphi\, c\psi \frac{\partial}{\partial\mu} - s\varphi\, s\psi \frac{\partial}{\partial\nu} \\
L_{14} &= -c\varphi\, s\psi \frac{\partial}{\partial\varphi} - s\varphi\, c\psi \frac{\partial}{\partial\psi} - s\varphi\, s\psi \frac{\partial}{\partial\mu} - c\varphi\, c\psi \frac{\partial}{\partial\nu} \\
L_{13} &= -c\varphi\, c\psi \frac{\partial}{\partial\varphi} + s\varphi\, s\psi \frac{\partial}{\partial\psi} - s\varphi\, c\psi \frac{\partial}{\partial\mu} + c\varphi\, s\psi \frac{\partial}{\partial\nu} \\
L_{24} &= s\varphi\, s\psi \frac{\partial}{\partial\varphi} - c\varphi\, c\psi \frac{\partial}{\partial\psi} - c\varphi\, s\psi \frac{\partial}{\partial\mu} + s\varphi\, c\psi \frac{\partial}{\partial\nu}.
\end{aligned}\right\} \tag{A 1}$$

The group $L(2,2)$ has p equal to q, and therefore the dimension of the subgroup A is the same as the number of invariants. According to section 2 they are apart from a factor

$$I^2 = -L_{12}^2 - L_{34}^2 + L_{23}^2 + L_{14}^2 + L_{13}^2 + L_{24}^2$$

$$I^{\frac{1}{2}} = L_{12} L_{34} - L_{13} L_{24} + L_{23} L_{14}.$$

Expressed in the realization of L_{ij} we get

$$I^2 = \left(\frac{\partial}{\partial \mu} + 1\right)^2 + \frac{\partial^2}{\partial \nu^2} - 1$$

$$I^{\frac{1}{2}} = \left(\frac{\partial}{\partial \mu} + 1\right) \frac{\partial}{\partial \nu}.$$

According to the results of section 3 the vectors $\tilde{f}$ in the representation space are functions of φ, ψ, μ and ν. In an irreducible space the dependence on μ and ν is fixed by putting for the vectors $\tilde{f}$

$$\tilde{f}(\varphi, \psi, \mu, \nu) = e^{i(a+i)\mu} e^{ib\nu} f(\varphi, \psi),$$

where a and b are arbitrary complex numbers. The irreducible space characterized by (a, b) corresponds to the invariants

$$I^2 = -a^2 - b^2 - 1$$

$$I^{\frac{1}{2}} = -ab$$

Already at this stage is it possible to obtain limitations on (a,b) due to unitarity, since in a unitary representation I^2 and $I^{\frac{1}{2}}$ have to be real. However we postpone this discussion till later on. Before determining the scalar product which defines the Hilbert space it is advantageous to change the parameters of K by the substitution

$$\varphi_1 = \varphi + \psi$$

$$\varphi_2 = \varphi - \psi,$$

and to choose new generators of the Lie-algebra corresponding to $L^{(1)}(1, 2)$ and $L^{(2)}(1, 2)$,

$$\left.\begin{aligned}
H_1 &\equiv \tfrac{1}{2}(L_{12} + L_{34}) = -\frac{\partial}{\partial \varphi_1} \\
H_2 &\equiv \tfrac{1}{2}(L_{12} - L_{34}) = -\frac{\partial}{\partial \varphi_2} \\
M_1 &\equiv -\tfrac{1}{2}(L_{23} - L_{14}) = -s\varphi_1 \frac{\partial}{\partial \varphi_1} - \left(i\frac{b-a}{2} + \frac{1}{2}\right) c\varphi_1 \\
M_2 &\equiv -\tfrac{1}{2}(L_{23} + L_{14}) = -s\varphi_2 \frac{\partial}{\partial \varphi_2} - \left(-i\frac{a+b}{2} + \frac{1}{2}\right) c\varphi_2 \\
N_1 &\equiv \tfrac{1}{2}(L_{13} + L_{24}) = -c\varphi_1 \frac{\partial}{\partial \varphi_1} + \left(i\frac{b-a}{2} + \frac{1}{2}\right) s\varphi_1 \\
N_2 &\equiv \tfrac{1}{2}(L_{13} - L_{24}) = -c\varphi_2 \frac{\partial}{\partial \varphi_2} + \left(-i\frac{b+a}{2} + \frac{1}{2}\right) s\varphi_2.
\end{aligned}\right\} \qquad (A\,2)$$

From these expressions we see that H_1, M_1, N_1 and H_2, M_2, N_2 both span the Lie-algebra of $L(1,2)$. The invariants of $L^{(1)}(1,2)$ and $L^{(2)}(1,2)$ are

$$I_1^2 = -H_1^2 + M_1^2 + N_1^2 = -\tfrac{1}{4}[(b-a)^2+1]$$

$$I_2^2 = -H_2^2 + M_2^2 + N_2^2 = -\tfrac{1}{4}[(b+a)^2+1],$$

which are two different combinations of I^2 and $I^{\frac{1}{2}}$. At this stage it may be appropriate to discuss the ranges of the angles φ, ψ and φ_1, φ_2. We noticed earlier that the ranges $0 \leqslant \varphi, \psi < 2\pi$ correspond to a parametrization of $L(2,2)$. Therefore the functions $f(\varphi,\psi)$ in the representation space should satisfy the continuity condition

$$f(\varphi,\psi) = f(\varphi+2\pi,\psi) = f(\varphi,\psi+2\pi)$$

if we look for representations of $L(2,2)$. This means that φ_1 and φ_2 take values in the intervals $(0,4\pi)$ and $(-2\pi,2\pi)$ respectively, so that we have to look for double-valued representations of $L^{(1)}(1,2)\otimes L^{(2)}(1,2)$ to find all representations of $L(2,2)$. This also follows from the fact that $L(2,2)$ contains a center of order 2 while $L^{(1)}(1,2)\otimes L^{(2)}(1,2)$ has no center, and thus $L(2,2)$ covers $L^{(1)}(1,2)\otimes L^{(2)}(1,2)$ twice. It is also clear that if one looks for the representations of the universal covering group $\bar{L}(2,2) \approx \bar{L}^{(1)}(1,2)\otimes \bar{L}^{(2)}(1,2)$ there is periodicity neither in φ nor in ψ.

From now on we just consider the group $L(1,2)$ since it is obvious how one can build the representations of $L(2,2)$ from those of $L(1,2)$. By putting

$$\left.\begin{aligned} H &= -\frac{\partial}{\partial\varphi} \\ M &= -s\varphi\frac{\partial}{\partial\varphi} - (ia+\tfrac{1}{2})\,c\varphi \\ N &= -c\varphi\frac{\partial}{\partial\varphi} + (ia+\tfrac{1}{2})\,s\varphi \end{aligned}\right\} \tag{A 3}$$

we now have to find a kernel $K(\varphi,\varphi')$ such that H, M and N are anti-Hermitean and such that

$$(f,g) \equiv \iint_0^{2\pi} \overline{f(\varphi)}\,K(\varphi,\varphi')\,g(\varphi')\,d\varphi\,d\varphi' \tag{A 4}$$

is a positive definite scalar product. From the anti-Hermiticity of H it follows that K depends only on $\varphi-\varphi'$. We now make the expansion

$$f(\varphi) = \sum_m f_m e^{im\varphi}, \tag{A 5}$$

where m takes the values $0, \pm 1, \pm 2, \ldots$ or $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$ depending on whether we look for single-valued or double-valued representations. A vector is now represented by a sequence $\{f_m\}$ and the operators of the Lie-algebra act on $\{f_m\}$ according to

$$\left.\begin{aligned} (Hf)_m &= -im\,f_m \\ (Mf)_m &= \left(-\frac{m}{2}-\frac{ia}{2}+\frac{1}{4}\right) f_{m-1} + \left(\frac{m}{2}-\frac{ia}{2}+\frac{1}{4}\right) f_{m+1} \\ (Nf)_m &= i\left(-\frac{m}{2}-i\frac{a}{2}+\frac{1}{4}\right) f_{m-1} - i\left(\frac{m}{2}-i\frac{a}{2}+\frac{1}{4}\right) f_{m+1}. \end{aligned}\right\} \tag{A 6}$$

The scalar product takes the form

$$(f,g) = \sum_m \bar{f}_m b_m g_m \tag{A 7}$$

where

$$b_m = \iint_0^{2\pi} e^{-im(\varphi-\varphi')} K(\varphi,\varphi')\,d\varphi\,d\varphi'. \tag{A 8}$$

The anti-Hermiticity of M (or N) now gives the following equations for the coefficients b_m

$$\left.\begin{aligned} (m - i\bar{a} + \tfrac{1}{2})\, b_{m+1} &= (m - ia + \tfrac{1}{2})\, b_m \\ (m + ia + \tfrac{1}{2})\, b_{m+1} &= (m + i\bar{a} + \tfrac{1}{2})\, b_m. \end{aligned}\right\} \tag{A 9}$$

By taking the difference of these equations one finds

$$(\bar{a}+a)\,b_{m+1} = (\bar{a}+a)\,b_m.$$

Therefore if $\bar{a}+a \neq 0$ we get $b_{m+1}=b_m$ and also from eqs. (A9) $\bar{a}=a$ i.e., a is real. From eq. (A8) one then finds that

$$K(\varphi,\varphi') = \text{Const}\,\delta(\varphi-\varphi'),$$

i.e., we have a one-point measure, and the representation thus defined belongs to the main series. If on the other hand $\bar{a}+a=0$, i.e. a is imaginary or zero, then one obtains the following recursion relation for b_m

$$b_{m+1} = b_m \frac{m+\frac{1}{2}+\alpha}{m+\frac{1}{2}-\alpha}, \tag{A 10}$$

where $\alpha = Ima$. The positive definiteness of the scalar product requires all b_m to be positive for all m-values, which appear in the sum (A7). This is possible only if $-\frac{1}{2}<\alpha<\frac{1}{2}$, when m takes all integer values. The sequences $\{f_m\}$ can however be bounded from below or above. According to eqs. (A6) one must then have

$$\alpha = -\underline{m} + \tfrac{1}{2}$$

and

$$\alpha = \bar{m} + \tfrac{1}{2}$$

respectively, where $\underline{m}$ and $\bar{m}$ are the lower and upper bounds for m. Then it is seen from eq. (A10) that all b_m are positive for $m \geq \underline{m} > 0$ and $m \leq \bar{m} < 0$ respectively. (It is also clear that $-\alpha$ defines the same representation as α, the difference is that now $b_m = 0$ for $m < \underline{m}$ or $> \bar{m}$ respectively.) We summarize the results in a table.

Single- and double-valued representations of $L(1,2)$

Range of a	Range of $I^2 = -H^2+M^2+N^2 = -a^2-\frac{1}{4}$	Spectrum of H	Type of representation
$-\infty < a < \infty$	$-\infty < I^2 \leqslant -\frac{1}{4}$	$0, \pm 1, \pm 2, \ldots$ or $\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$	Main series; local scalar product.
$-\frac{i}{2} < a < \frac{i}{2}$	$-\frac{1}{4} \leqslant I^2 < 0$	$0, \pm 1, \pm 2, \ldots$	Continuous part of supplementary series; non-local scalar product.
$a = \pm i(\underline{m}-\frac{1}{2})$ $\underline{m}$ integer or half-integer $\geqslant \frac{1}{2}$	$I^2 = \underline{m}(\underline{m}-1)$	$\underline{m}, \underline{m}+1, \ldots$	Discrete part of supplementary series, bounded below; non-local scalar product.
$a = \pm i(\bar{m}+\frac{1}{2})$ $\bar{m}$ integer or half-integer $\leqslant -\frac{1}{2}$	$I^2 = \bar{m}(\bar{m}+1)$	$\bar{m}, \bar{m}-1, \ldots$	Discrete part of supplementary series, bounded above; non-local scalar product.

These results may be compared with those of Bargmann [4], and we find that we have obtained all irreducible unitary representations. It is seen that such a representation is characterized by the value of the invariant and the spectrum of H. The value of the invariant is thus not sufficient alone, and the parameter a has no advantage in that respect. The two values $\pm a$ define unitary equivalent representations.

ACKNOWLEDGEMENT

This work was carried out while the author was a visiting scientist at Cern. I want to thank professor L. van Hove for the hospitality extended to me there. I would also like to thank professor N. Svartholm for his kind interest in my work. Finally a grant from the Swedish Atomic Research Council is acknowledged.

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Tryckt den 1 oktober 1965

Uppsala 1965. Almqvist & Wiksells Boktryckeri AB

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On the unitary irreducible representations of the pseudo-orthogonal group $L(3,3)$

ALMQVIST & WIKSELL
STOCKHOLM
GÖTEBORG · UPPSALA
1966

ARKIV FÖR FYSIK Band 32 nr 13

3.66 03 5 Communicated 9 February 1966 by L. HULTHÉN and N. SVARTHOLM

On the unitary irreducible representations of the pseudo-orthogonal group $L(3,3)$

By ARNE KIHLBERG

ABSTRACT

The pseudo-orthogonal group which conserves the quadratic form

$$x_1^2+x_2^2+x_3^2-x_4^2-x_5^2-x_6^2$$

together with its Lie-algebra is studied. A number of series of unitary irreducible representations of the universal covering group is derived with a Lie-algebra method, whose justification leans heavily on works by Harish-Chandra.

Introduction

One of the most urgent problems in elementary particle physics today is the explanation of the different states, elementary particles or resonances, in which matter can appear. For several years one has tried intensively to bring some order into the relations betwen these states by means of group theory. The group which has been studied most is $SU(3)$ [1]. It has been assumed that the strongest interactions are invariant under the transformations of $SU(3)$ and that the particles or resonances should be grouped into multiplets corresponding to unitary irreducible representations of $SU(3)$.

However, since the predictions of the $SU(3)$-theory have been moderately successful only, one has also considered the possibility of finding other groups which would better reflect Nature. In this research one has also encountered non-compact groups. Since a non-compact group has no faithful, unitary, finite-dimensional representation one has either to give up unitarity or to consider infinite particle multiplets. Both possiblities involve certain complications which we will not discuss here. However, one should take notice of a change in attitude towards the problem in several recent papers [2]. It is suggested that the non-compact group or the corresponding Lie-algebra should not necessarily express a symmetry of the interaction but that it still could be useful to generate the spectra of mass, spin, isospin, hypercharge and baryon number operators.

In this paper we shall derive a number of series of unitary, irreducible representations of the universal covering group $\bar{L}(3,3)$ of the pseudo-orthogonal group

$L(3,3)$. More precisely $L(3,3)$ denotes the component connected to the identity of the group of real linear transformations which conserve the quadratic form

$$x_1^2+x_2^2+x_3^2-x_4^2-x_5^2-x_6^2.$$

The physical interpretation is left to a fourthcoming paper. We shall, however, briefly outline the origin of this group in a geometrical model which we have proposed earlier [3].

In ref. [3] we defined an 8-dimensional reference system of the following nature based on the properties of the photon. Four of the coordinates denoted x_1, x_2, x_3 and t refer to the ordinary space-time manifold and, in addition, at each point in spacetime there are defined two three-dimensional orthogonal vectors $\mathbf{e}_0$ and $\mathbf{h}_0$ of equal length. These vectors are assumed to be independent of the space-time point to which they belong. Furthermore, we assume that an elementary particle may be described in terms of eight coordinates, four of which are (x_1, x_2, x_3, t). The other four coordinates can be described by two vectors $\mathbf{e}$ and $\mathbf{h}$ of the same nature as $\mathbf{e}_0$ and $\mathbf{h}_0$. Instead of $\mathbf{e}$ and $\mathbf{h}$ one may use the three Euler angles (φ, θ, ψ), which define the relative orientation of $(\mathbf{e}, \mathbf{h})$ with regard to $(\mathbf{e}_0, \mathbf{h}_0)$, and the scale coordinate $s=\ln|\mathbf{e}|/|\mathbf{e}_0|$. In ref. [3] we also discussed the problem of finding the appropriate symmetry group of this space. First of all, it seems most natural to require invariance under the Poincaré group P. P acts both on the space coordinates and on the vectors $(\mathbf{e}_0, \mathbf{h}_0)$. The infinitesimal generators for time and space translations, rotations and accelerations can be written

$$\begin{aligned}
P_t &= \frac{\partial}{\partial t}\\
P_i &= \frac{\partial}{\partial x_i}, \quad i=1,2,3\\
M_i &= -\varepsilon_{ijk}\, x_j \frac{\partial}{\partial x_k} + S_i\\
M_i' &= t\frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial t} + S_i'.
\end{aligned} \tag{1}$$

The operators S_i and S_i' act on the vectors $(\mathbf{e}_0, \mathbf{h}_0)$ or, equivalently, on the variables φ, θ, ψ and s. For these one has the expressions

$$\begin{aligned}
S_1 &= c\varphi \frac{c\theta}{s\theta}\frac{\partial}{\partial\varphi} + s\varphi\frac{\partial}{\partial\theta} - \frac{c\varphi}{s\theta}\frac{\partial}{\partial\psi}\\
S_2 &= s\varphi \frac{c\theta}{s\theta}\frac{\partial}{\partial\varphi} - c\varphi\frac{\partial}{\partial\theta} - \frac{s\varphi}{s\theta}\frac{\partial}{\partial\psi}\\
S_3 &= -\frac{\partial}{\partial\varphi}
\end{aligned}$$

$$S_1' = \frac{s\varphi}{s\theta}\frac{\partial}{\partial\varphi} - c\varphi\, c\theta\,\frac{\partial}{\partial\theta} - s\varphi\,\frac{c\theta}{s\theta}\frac{\partial}{\partial\psi} - c\varphi\, s\theta\left(\frac{\partial}{\partial s} - 1\right)$$

$$S_2' = -\frac{c\varphi}{s\theta}\frac{\partial}{\partial\varphi} - s\varphi\, c\theta\,\frac{\partial}{\partial\theta} + c\varphi\,\frac{c\theta}{s\theta}\frac{\partial}{\partial\psi} - s\varphi\, s\theta\left(\frac{\partial}{\partial s} - 1\right)$$

$$S_3' = s\theta\,\frac{\partial}{\partial\theta} - c\theta\left(\frac{\partial}{\partial s} - 1\right). \tag{2}$$

In addition to the transformations of eq. (1) we now assume that a rotation of the particle around the internal axis is also a symmetry operation. The corresponding infinitesimal generators are

$$T_1 = -\frac{c\psi}{s\theta}\frac{\partial}{\partial\varphi} + s\psi\frac{\partial}{\partial\theta} + c\psi\,\frac{c\theta}{s\theta}\frac{\partial}{\partial\psi}$$

$$T_2 = -\frac{s\psi}{s\theta}\frac{\partial}{\partial\varphi} - c\psi\frac{\partial}{\partial\theta} + s\psi\,\frac{c\theta}{s\theta}\frac{\partial}{\partial\psi}$$

$$T_3 = -\frac{\partial}{\partial\psi}. \tag{3}$$

It should be noted that the invariance under the transformations of eq. (1) is motivated by relativity while the invariance under the transformations of eq. (3) requires some kind of spherical symmetry of the particle. Whether or not this is the case we shall assume that the generators of eqs. (1) and (3) form part of the Lie-algebra of a Lie-group. The remaining generators are obtained by means of commutation. We anticipate the result and introduce the following notations

$$\begin{aligned}
S_1 &= L_{23}\\
S_2 &= -L_{13}\\
S_3 &= L_{12}\\
S_1' &= L_{14}\\
S_2 &= L_{24}\\
S_3' &= L_{34}\\
T_1 &= L_{45}\\
T_2 &= -L_{46}\\
T_3 &= L_{56}.
\end{aligned} \tag{4}$$

Then we find the explicit form of the additional generators from the appropriate commutators

$$L_{35}\equiv[M_3', T_1] = c\psi\,\frac{c\theta}{s\theta}\,\frac{\partial}{\partial\varphi} - s\psi\,c\theta\,\frac{\partial}{\partial\theta} - \frac{c\psi}{s\theta}\,\frac{\partial}{\partial\psi} - s\psi\,s\theta\left(\frac{\partial}{\partial s}-1\right)$$

$$L_{36}\equiv[T_2, M_3'] = -s\psi\,\frac{c\theta}{s\theta}\,\frac{\partial}{\partial\varphi} - c\psi\,c\theta\,\frac{\partial}{\partial\theta} + \frac{s\psi}{s\theta}\,\frac{\partial}{\partial\psi} - c\psi\,s\theta\left(\frac{\partial}{\partial s}-1\right)$$

$$L_{15}\equiv[M_1', T_1] = c\varphi\,c\psi\,\frac{\partial}{\partial\varphi} - c\varphi\,s\theta\,s\psi\,\frac{\partial}{\partial\theta} - s\varphi\,s\psi\,\frac{\partial}{\partial\psi} + (s\varphi\,c\psi + c\varphi\,c\theta\,s\psi)\left(\frac{\partial}{\partial s}-1\right)$$

$$L_{25}\equiv[M_2', T_1] = s\varphi\,c\psi\,\frac{\partial}{\partial\varphi} - s\varphi\,s\theta\,s\psi\,\frac{\partial}{\partial\theta} + c\varphi\,s\psi\,\frac{\partial}{\partial\psi} - (c\varphi\,c\psi - s\varphi\,c\theta\,s\psi)\left(\frac{\partial}{\partial s}-1\right)$$

$$L_{16}\equiv[T_2, M_1'] = -c\varphi\,s\psi\,\frac{\partial}{\partial\psi} - c\varphi\,s\theta\,c\psi\,\frac{\partial}{\partial\theta} - s\varphi\,c\psi\,\frac{\partial}{\partial\psi} - (s\varphi\,s\psi - c\varphi\,c\theta\,c\psi)\left(\frac{\partial}{\partial s}-1\right)$$

$$L_{26}\equiv[T_2, M_1'] = -s\varphi\,s\psi\,\frac{\partial}{\partial\varphi} - s\varphi\,s\theta\,c\psi\,\frac{\partial}{\partial\theta} + c\varphi\,c\psi\,\frac{\partial}{\partial\psi} + (c\varphi\,s\psi + s\varphi\,c\theta\,c\psi)\left(\frac{\partial}{\partial s}-1\right). \tag{5}$$

These six generators together with the operators M_i' generate all the operators S_i and T_i. Therefore the generators $L_{\mu\nu}=-L_{\nu\mu}$ for $\mu,\nu=1\ldots6$ span a subalgebra. Its commutation relations are

$$[L_{\mu\nu}, L_{\varrho\sigma}] = \gamma_{\nu\varrho}, L_{\mu\sigma} + \gamma_{\mu\sigma}, L_{\nu\varrho} - \gamma_{\mu\varrho}, L_{\nu\sigma} - \gamma_{\nu\sigma}, L_{\mu\varrho}, \tag{6}$$

where γ is the matrix

$$\gamma = \begin{bmatrix} -1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}.$$

Thus the generators $L_{\mu\nu}$ span the Lie-algebra of the pseudo-orthogonal group $L(3,3)$. By subtraction of S_i and S_i' from M_i and M_i' in the eq. (1) we find that the Lie-algebra obtained by starting from the generators of the eqs. (1) and (3) is the direct sum of the Lie-algebra of the Poincaré group and the Lie-algebra of $L(3,3)$. The corresponding group is not necessarily the direct product of P and $L(3,3)$, however in quantum mechanics one is interested in the universal covering group and the representations of this group are in one-to-one correspondence with the representations of the Lie-algebra. Therefore we shall in the following limit our attention to the Lie-algebra and in fact to that of $L(3,3)$ since the unitary representations of the Lie-algebra of P are well known.

The physical discussion in a forthcoming paper will include a discussion of the difference between the transformations of eq. (1) and eq. (3). If the transformations of eq. (3) are assumed to be symmetry transformations of an interaction, then the particles must be " spherically symmetric". If it is assumed that the group is just a "spectrum generating group" we are not forced to restrict the attention only to unitary representations, but we shall always consider only unitary representations.

Some results regarding the problem of finding the irreducible unitary representations of $\bar{L}(3,3)$ are known in the mathematical literature. Gelfand and Graev [4] have given three fundamental series of representations of the group $SL[4, R)$, the group of real, 4×4, unimodular matrices. $SL(4, R)$ is locally isomorphic to $L(3,3)$ but it is neither isomorphic to $L(3,3)$ nor to $\bar{L}(3,3)$ since it has a centre of order two while $L(3,3)$ has no centre and $\bar{L}(3,3)$ has a centre of order four. Graev and Gelfand use a very elaborate technique and in our opinion the form of the representations is not well suited for a physical interpretation. In this paper we shall apply the method of reference [5]. This method is convenient to apply to all semi-simple groups as long as the order of the group is not too high.

2. A realization of the Lie-algebra based on the Iwasawa decomposition

The Iwasawa decomposition states that any semi-simple connected Lie-group can be written as a product of three subgroups [5]

$$L = KAN. \tag{7}$$

Here K is the maximal compact subgroup of L, that is $SO_3 \otimes SO_3$ in the case of $L(3,3)$. A is Abelian and in our case it is generated by the operators L_{34}, L_{25} and L_{16}. The subgroup N, finally, is solvable and for $L(3,3)$ its generators may be chosen as

$$\begin{aligned} N_1 &= L_{14} - L_{13} \\ N_2 &= L_{24} - L_{23} \\ N_3 &= L_{35} - L_{45} \\ N_4 &= L_{36} - L_{46} \\ N_5 &= L_{26} - L_{56} \\ N_6 &= L_{15} - L_{12}. \end{aligned} \tag{8}$$

In passing we note that N is an invariant subgroup of the group AN. It is convenient to parametrize the group in the following way

$$\begin{aligned} g = {} & e^{\varphi L_{12}} e^{-\theta L_{13}} e^{\psi L_{12}} e^{\alpha L_{56}} e^{-\beta L_{46}} e^{\gamma L_{56}}, \\ & e^{\lambda L_{34}} e^{\mu L_{25}} e^{\nu L_{16}}, \\ & e^{q N_1} e^{r N_2} e^{s N_3} e^{t N_4} e^{u N_5} e^{v N_6}. \end{aligned} \tag{9}$$

where g is an arbitrary element of the group.

A realization of the group operations can now be given as transformations on the group parameters induced by left multiplication. However, it is sufficient to consider the transformations of the parameters of K and A only [5]. In particular we obtain in this way a realization of the infinitesimal generators $L_{\mu\nu}$ as differential operators on the group spaces of K and A.

$$
\begin{aligned}
L_{23} &= c\varphi \frac{c\theta}{s\theta}\frac{\partial}{\partial\varphi} + s\varphi\frac{\partial}{\partial\theta} - \frac{c\varphi}{s\theta}\frac{\partial}{\partial\psi} \\
L_{13} &= -s\varphi \frac{c\theta}{s\theta}\frac{\partial}{\partial\varphi} + c\varphi\frac{\partial}{\partial\theta} + \frac{s\varphi}{s\theta}\frac{\partial}{\partial\psi} \\
L_{12} &= -\frac{\partial}{\partial\varphi} \\
L_{45} &= c\alpha \frac{c\beta}{s\beta}\frac{\partial}{\partial\alpha} + s\alpha\frac{\partial}{\partial\beta} - \frac{c\alpha}{s\beta}\frac{\partial}{\partial\gamma} \\
L_{46} &= -s\alpha \frac{c\beta}{s\beta}\frac{\partial}{\partial\alpha} + c\alpha\frac{\partial}{\partial\beta} + \frac{s\alpha}{s\beta}\frac{\partial}{\partial\gamma} \\
L_{56} &= -\frac{\partial}{\partial\alpha} \\
L_{34} &= s\theta\, c\beta \frac{\partial}{\partial\theta} - s\theta\, c\psi\, s\beta\, s\gamma \frac{\partial}{\partial\psi} + c\theta\, s\beta\frac{\partial}{\partial\beta} - s\theta\, s\psi\, s\beta\, c\gamma\frac{\partial}{\partial\gamma} \\
&\quad + c\theta\, c\beta\frac{\partial}{\partial\lambda} + s\theta\, s\psi\, s\beta\, s\gamma\frac{\partial}{\partial\mu} - s\theta\, c\psi\, s\beta\, c\gamma\frac{\partial}{\partial\nu}.
\end{aligned}
\tag{10}
$$

The remaining eight generators are most easily obtained by means of the commutation relations. The realization (10) may be compared to the realization of the eqs. (2), (3), (5). In the latter case the carrier space is four-dimensional whereas the realization (10) is based on a nine-dimensional carrier space. In principle, the realization (10) will allow us to obtain all unitary irreducible representations, whereas the other realization only admits some of these representations [6].

The enveloping algebra of the Lie-algebra is of some help for the classification of the irreducible representations. The group $L(3,3)$ has the following three invariants which then generate the centre of the enveloping algebra [5]

$$
\begin{aligned}
I_2 &= \gamma^{\mu\varrho}\gamma^{\nu\sigma} L_{\mu\nu} L_{\varrho\sigma} \equiv L_{\mu\nu} L^{\mu\nu} \\
I_4 &= L_{\mu\nu} L^{\nu\varrho} L_{\varrho\sigma} L^{\sigma\mu} \\
I_{\frac{1}{2}} &= \varepsilon^{\mu\nu\varrho\sigma\tau\pi} L_{\mu\nu} L_{\varrho\sigma} L_{\tau\pi}.
\end{aligned}
\tag{11}
$$

After the insertion of the explicit expressions (10) for $L_{\mu\nu}$ and some rather long calculations one obtains[1]

[1] I am grateful to prof. N. Svartholm for some lemmas which facilitate the calculations considerably. (Internal report of the Institute of theoretical physics, Göteborg).

$$I_2 = -2\left[\left(\frac{\partial}{\partial\gamma}-2\right)^2+\left(\frac{\partial}{\partial\mu}-1\right)^2+\frac{\partial^2}{\partial\nu^2}+5\right]$$

$$I_4 = 2\left[\left(\frac{\partial}{\partial\lambda}-2\right)^4+\left(\frac{\partial}{\partial\mu}-1\right)^4+\frac{\partial^4}{\partial\nu^4}\right]$$

$$-4\left[\left(\frac{\partial}{\partial\lambda}-2\right)^2+\left(\frac{\partial}{\partial\mu}-1\right)^2+\frac{\partial^2}{\partial\nu^2}\right]-14$$

$$I_{\frac{1}{2}} = 48\left(\frac{\partial}{\partial\lambda}-2\right)\left(\frac{\partial}{\partial\mu}-1\right)\frac{\partial}{\partial\nu}. \tag{12}$$

Note that λ, μ and ν are the parameters of the subgroup A.

3. Choice of representation space and the determination of the scalar product

In the realization (10) of the Lie-algebra it is tacitly assumed that the space on which the elements of the Lie-algebra act is a linear space of differentiable functions on the topological product $K \times A$. Now this space has to be somewhat more specified. It is clear that we shall have to introduce a scalar product in order to obtain a Hilbert space. However, before doing this we shall specify the dependence of the functions an the parameters of A. This is possible since the coefficients of the derivatives in a generator $L_{\mu\nu}$ do not depend on λ, μ and ν. Or expressed in another way, the operators $\partial/\partial\lambda$, $\partial/\partial\mu$ and $\partial/\partial\nu$ commute with the Lie-algebra and, therefore, we can choose elements in the representation space which are eigenfunctions of $\partial/\partial\lambda, \partial/\partial\mu$ and $\partial/\partial\nu$. The eigenvalues are denoted $(ia+2)$, $(ib+1)$ and ic respectively where a, b and c are complex numbers. Thus, a general element in the representation space has the form

$$\tilde{f}(\varphi, \theta, \ldots, \lambda, \mu, \nu) = e^{i\lambda(b-2i)}\, e^{i\mu(b-i)}\, e^{i\nu c} f(\varphi, \theta, \psi, \alpha, \beta, \gamma), \tag{13}$$

where f is a function on K. The values of the invariants in this space characterized by a, b and c are

$$I_2 = 2\,(a^2+b^2+c^2+5)$$

$$I_4 = 2\,(a^4+b^4+c^4)+4\,(a^2+b^2+c^2)-14$$

$$I_{\frac{1}{2}} = 48\ iabc. \tag{14}$$

In an irreducible representation the invariants have constant values. Furthermore, in a unitary representation I_2 and I_4 must be real and $I_{\frac{1}{2}}$ imaginary. The restrictions on a, b and c which follow from this requirement are not sufficient to determine the allowed values of a, b and c. This has to be done by introducing the scalar product which makes the generators $L_{\mu\nu}$ anti-Hermitean, In ref. [5] it was shown that this requirement implies that the scalar product can be written

$$(\bar{f}_1, \bar{f}_2) = \iint_{K \times K} \bar{f}_1(k_1)\, M(k_1, k_2)\, f_2(k_2)\, dk_1\, dk_2, \tag{15}$$

where k stands for a point in the group space of K, dk is the invariant measure on K, and where the kernel M has to satisfy

$$\begin{aligned} \bar{M}(k_1, k_2) &= M(k_2, k_1) \\ M(k_1 k, k_2 k) &= M(k_1, k_2), \end{aligned} \tag{16}$$

i.e. M is an invariant hermitean two-point measure on K. The relations (16) follow from the anti-hermiticity conditions of the compact operators only. To determine M completely one has to require that also L_{34} is anti-hermitean and, furthermore, that M is a positive definite kernel. L_{34} contains the parameters a, b and c and, therefore, we obtain restrictions on these at the same time.

Instead of working with the functions $f(\varphi, \theta, \ldots, \gamma)$ and the realization (10) of the generators it is at this stage advantageous to introduce a basis i.e. a suitable function system on K or rather $\bar{K} = \overline{SO}_3 \otimes \overline{SO}_3$ since we are looking for representations of $\bar{L}$. A natural choice is at hand, namely, the functions

$$T^l_{mn}(\varphi, \theta, \psi)\, D^j_{hk}(\alpha, \beta, \gamma), \tag{17}$$

where

$$\begin{gathered} 2l = 0, 1, 2, \ldots, \quad -l \leqslant m, n \leqslant l \\ 2j = 0, 1, 2, \ldots, \quad -j \leqslant h, k \leqslant j \\ 0 \leqslant \theta, \beta \leqslant \pi, \quad 0 \leqslant \begin{matrix}\varphi+\psi \\ \alpha+\gamma\end{matrix} \leqslant 4\pi, \; -2\pi \leqslant \begin{matrix}\varphi-\psi \\ \alpha-\gamma\end{matrix} \leqslant 2\pi. \end{gathered}$$

T^l_{mn} denotes the matrix element in an irreducibie representation l, of a finite transformation of $\overline{SO}_3$ described by the Euler angles φ, θ and ψ. We know that the set $\{T^l_{mn}\}$ is a complete orthogonal set of functions on $\overline{SO}_3$ with respect to the invariant measure on $\overline{SO}_3$. Similar properties hold for $\{D^j_{hk}\}$. The defining relations and various properties of these functions are given in an appendix. Let us now represent an arbitrary function $f(\varphi, \theta, \psi, \alpha, \beta, \gamma)$ by the sequence $\{f(lmnjhk)\}$ according to the expansion

$$f = \sum_{(lmnjhk)} f(lmnjhk)\, T^l_{mn}\, D^j_{hk}. \tag{18}$$

The operators $L_{\mu\nu}$ can now be considered as operators acting on the indices $(lmn\,jhk)$. The explicit formulae are easily obtained from eqs. (A4) of the appendix. The scalar product (15) takes the form

$$\begin{aligned} (\bar{f}_1, \bar{f}_2) = \sum_{\substack{(l_1 m_1 n_1 j_1 h_1 k_1) \\ (l_2 m_2 n_2 j_2 h_2 k_2)}} & \bar{f}_1(l_1 m_1 n_1 j_1 h_1 k_1) \\ & \times M(l_1 m_1 n_1 j_1 h_1 k_1 \,|\, l_2 m_2 n_2 j_2 h_2 k_2)\, f_2(l_2 m_2 n_2 j_2 h_2 k_2) \end{aligned} \tag{19}$$

where the summation is extended over suitable ranges which will be determined when the question of irreducibility is discussed. The positive definite hermitean matrix M has to be determined so that the $L_{\mu\nu}$'s are anti-hermitean. From the requirement that L_{12} and L_{56} are anti-hermitean it follows that M is nonzero only for $m_1 = m_2$ and $h_1 = h_2$. The anti-hermiticity of L_{13} and L_{46} then yields

$$M(lmnjhk\,|\,l'\,m'\,n'\,j'\,h'\,k') = (2\,l+1)^{-1}(2\,j+1)^{-1}\delta_{ll'}\,\delta_{jj'}\,\delta_{mm'}\,\delta_{hh'}\,\mathfrak{M}_{lj}(nk\,|\,n'\,k'). \qquad (20)$$

This formula could have been obtained also from eq. (16). The reduced matrix $\mathfrak{M}_{lj}(nk\,|\,n'\,k')$ has to be determined from the antihermiticity of L_{34}. The relations obtained from this requirement are given by the eqs. (A5, A6, A7, A8, A9, A10).

We can now discuss the different series of irreducible representations on the basis of these equations. It was noted earlier that an irreducible space is characterized by the values of the parameters a, b and c. From the eqs. (A4, ..., A10) it can be seen that the Hilbert space which is spanned by all functions $f(lmn\ jhk)$ and for which the scalar product is given by eq. (20) decomposes into eight invariant subspaces $H_{2\varrho_0 2\tau_0}$ where $2\varrho_0 = n + k$ mod. 2 and $2\tau_0 = n - k$ mod. 2. They are listed in table 1.

Table 1

Invariant subspace	Values of (n, k)	1	j
H_{00}	$\ldots(0,0),\ (1,1),\ (1,-1),\ldots$	$0, \pm1, \pm2\ldots$	$0, \pm1, \pm2\ldots$
H_{01}	$\ldots(-\frac{1}{2},\frac{1}{2})\ (\frac{1}{2},\frac{3}{2}),\ (\frac{1}{2},-\frac{1}{2}),\ldots$	$\pm\frac{1}{2}, \pm\frac{3}{2}\ldots$	$\pm\frac{1}{2}, \pm\frac{3}{2}\ldots$
H_{10}	$\ldots(-\frac{1}{2},-\frac{1}{2}),\ (\frac{1}{2},\frac{1}{2}),\ (\frac{1}{2},-\frac{3}{2})\ldots$	$\pm\frac{1}{2}, \pm\frac{3}{2}\ldots$	$\pm\frac{1}{2}, \pm\frac{3}{2}\ldots$
H_{11}	$\ldots(0,-1),\ (1,0),\ (1,-2)\ldots$	$0, \pm1, \pm2\ldots$	$0, \pm1, \pm2\ldots$
$H_{\frac{1}{2}\frac{1}{2}}$	$\ldots(-\frac{1}{2},1),\ (\frac{1}{2},2),\ (\frac{1}{2},0)\ldots$	$\pm\frac{1}{2}, \pm\frac{3}{2}\ldots$	$0, \pm1, \pm2\ldots$
$H_{\frac{1}{2}\frac{3}{2}}$	$\ldots(0,\frac{1}{2}),\ (1,\frac{3}{2}),\ (1,-\frac{1}{2})\ldots$	$0, \pm1, \pm2\ldots$	$\pm\frac{1}{2}, \pm\frac{3}{2}\ldots$
$H_{\frac{3}{2}\frac{1}{2}}$	$\ldots(0,-\frac{1}{2}),\ (1,\frac{1}{2}),\ (1,-\frac{3}{2})\ldots$	$0, \pm1, \pm2\ldots$	$\pm\frac{1}{2}, \pm\frac{3}{2}\ldots$
$H_{\frac{3}{2}\frac{3}{2}}$	$\ldots(-\frac{1}{2},0),\ (\frac{1}{2},1),\ (\frac{1}{2},-1)\ldots$	$\pm\frac{1}{2}, \pm\frac{3}{2}\ldots$	$0, \pm1, \pm2\ldots$

The representation induced in the subspace $H_{2\varrho_0 2\tau_0}$ and characterized by the parameters a, b and c will be denoted $V(\varrho_0, \tau_0, a, b, c)$. To classify the different series of irreducible unitary representations one has to find the values of a, b and c for which the eqs. (A5, ..., A10) have a solution with a positive definite matrix $\mathfrak{M}_{lj}(nk\,|\,n'\,k')$. In general this will be a complicated task because the equations are difference equations with many terms, However, if one looks for solutions of the form

$$\mathfrak{M}_{lj}(nk\,|\,n'\,k') = B_{ljnk}\,\delta_{nn'}\,\delta_{kk'} \qquad (21)$$

and puts

$$\begin{aligned} a_1 &= \mathrm{Re}\, a \\ a_2 &= \mathrm{Im}\, a \\ e_1 &= \mathrm{Re}\,(b+c) \\ e_2 &= \mathrm{Im}\,(b+c) \\ d_1 &= \mathrm{Re}\,(b-c) \\ d_2 &= \mathrm{Im}\,(b-c) \end{aligned} \tag{22}$$

then one finds from eq. (A10)

$$\begin{aligned} a_2\, nk\, B_{ljnk} &= 0 \\ (n+k+1-e_2)\, B_{ljn+1k+1} &= (n+k+1+e_2)\, B_{ljnk} \\ (n-k+1-d_2)\, B_{ljn+1k-1} &= (n-k+1+d_2)\, B_{ljnk} \\ e_1\, B_{ljn+1k+1} &= e_1\, B_{ljnk} \\ d_1\, B_{ljn+1k-1} &= d_1\, B_{ljnk}. \end{aligned} \tag{23}$$

Let us divide the discussion of these equations into two cases according to whether B_{ljnk} is different from zero only for $(n, k) = (0, 0)$ or other (n, k) values also appear. It is easily seen that one can have limitations on $n+k$ or $n-k$ but not on n or k separately. Therefore the second case implies $a_2 = 0$ according to the first of the eqs. (23).

Case I. (n, k) *not limited to* $(0, 0)$

Besides the eqs. (23) there are also the following relations which are obtained from the eqs. (A6, ..., A9)

$$\begin{aligned} B_{l+1j+1nk} &= B_{ljnk} \\ k\, B_{l+1jnk} &= k\, B_{ljnk} \\ n\, B_{lj+1nk} &= n\, B_{ljnk}. \end{aligned} \tag{24}$$

Therefore, B_{ljnk} is independent of both l and j except possibly for $n=0$ or $k=0$. However, one cannot have a limitation to $n=0$ separately and since the eqs. (23) allow us to determine B_{ljok} from some B_{lj1k} and B_{lj1k} is independent of j it follows that also B_{ljok} is independent of j. The same is true for l. Thus we see that as long as (n, k) is not limited to $(0, 0)$ the measure B_{ljnk} is independent of l and j and all (l, j) values appear which belong to a given subspace $H_{2\varrho_0 2\tau_0}$.

Let us now examine the dependence of B_{ljnk} on n and k. This dependence is given by the last four of the eqs. (23). We see that the equations containing $n+k$ are separated from those containing $n-k$. It is therefore possible to study just two equations at a time (put $B_{ljnk} = F_{\varrho\tau}$, $2\varrho = n+k$, $2\tau = n-k$).

$$\begin{aligned} (2\varrho+1-e_2)\, F_{\varrho+1\tau} &= (2\varrho+1+e_2)\, F_{\varrho\tau} \\ e_1\, F_{\varrho+1\tau} &= e_1\, F_{\varrho\tau}. \end{aligned} \tag{25}$$

One can first distinguish the case $e_1 \neq 0$. It then follows that $e_2 = 0$ and that $F_{\varrho\tau}$ does not depend on ϱ. Furthermore, all ϱ:s belonging to a given space $H_{2\varrho_0 2\tau_0}$

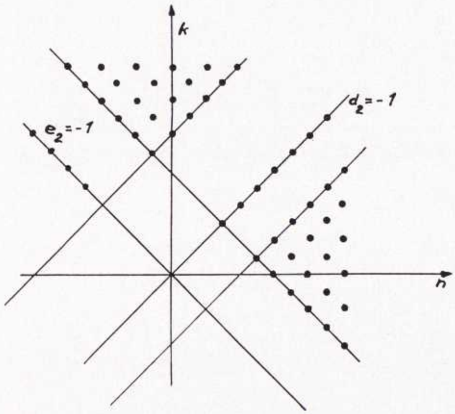

Fig. 1

appear. If $e_1 = 0$ then e_2 can be different from zero. Furthermore, if e_2 is sufficiently small one still has the full range of ϱ-values but $F_{\varrho\tau}$ will depend on ϱ. The limitations are

$$\begin{aligned} |e_2| &< 1 \quad \text{if} \quad 2\varrho_0 = 0 \\ |e_2| &= 0 \quad \text{if} \quad 2\varrho_0 = 1 \\ |e_2| &< \tfrac{1}{2} \quad \text{if} \quad 2\varrho_0 = \tfrac{1}{2} \\ |e_2| &< \tfrac{1}{2} \quad \text{if} \quad 2\varrho_0 = \tfrac{3}{3}. \end{aligned} \tag{26}$$

There is also the possibility of limitations on ϱ. First of all there is the possible solution $F_{\varrho\tau} = 0$ unless $\varrho = 0$. This requires $e_2 = -1$. But one can also have a lower positive bound $\underline{\varrho}$ or upper negative bound $\bar{\varrho}$ for ϱ. The value of e_2 is then given by

$$e_2 = -1 + 2\underline{\varrho}$$

or

$$e_2 = -1 - 2\bar{\varrho}. \tag{27}$$

The corresponding possibilities come out for the variables τ, d_1 and d_2 by using the other two equations (23). Some typical diagrams for the allowed (n, k) values are shown in fig. 1. Besides these restrictions on n and k there is, of course, always the conditions $|n| \leqslant l$, $|k| \leqslant j$. The explicit dependence of B_{ljnk} on n and k can, of course, be obtained from the eqs. (23). We do not give the expressions but we observe that for $e_1 \neq 0$ or B_{ljnk} independent of all its indices the corresponding measure $M(k_1, k_2)$ in eq. (15) is an ordinary one-point measure. We have previously called this series the main series [5].

Case II. Degenerate series

In dealing with case I we excluded the possibility that $(n, k) \equiv (0, 0)$, i.e., B_{ljnk} is different from zero only for $n = k = 0$. It can be included among some degenerate series obtained in the following way. Consider the equations (A4). By putting $n=k=0,\ 1+ib=0,\ c=0$ and

$$T^l_{mo}\,D^j_{ho}\equiv|\,ljmh\rangle \tag{28}$$

the action of the operator L_{34} is given by

$$\begin{aligned}
&L_{34}|\,ljmh\rangle\\
&=(2+ia+l+j)\frac{\sqrt{(l+m+1)\,(l-m+1)\,(j+h+1)\,(j-h+1)}}{(2l+1)\,(2j+1)}|\,l+1\,j+1\,mh\rangle\\
&\quad+(1+ia+l-j)\frac{\sqrt{(l+m+1)\,(l-m+1)\,(j+h)\,(j-h)}}{(2l+1)\,(2j+1)}|\,l+1\,j-1\,mh\rangle\\
&\quad+(1+ia-l+j)\frac{\sqrt{(l+m)\,(l-m)\,(j+h+1)\,(j-h+1)}}{(2l+1)\,(2j+1)}|\,l-1\,j+1\,mh\rangle\\
&\quad+(ia-l-j)\frac{\sqrt{(l+m)\,(l-m)\,(j+h)\,(j-h)}}{(2l+1)\,(2j+1)}|\,l-1\,j-1\,mh\rangle.
\end{aligned} \tag{29}$$

Together with the expressions for the compact operators we thus have a realization of the Lie-algebra on vectors which span representation spaces of the compact subgroup. Furthermore, there is nothing to prevent us from using this realization also for half-integer values of l and j. But it is then known from some general theorems on analytic vectors [7] that the algebraically irreducible hermitean representations of the Lie-algebra correspond to irreducible, unitary representations of the universal covering group. As a matter of fact these theorems form the basis for all our calculations.

The measure matrix is now defined as in eq. (20) with the omission of the indices n, k, n', k'. The reduced matrix $\mathcal{M}$ is therefore diagonal and depends only on l and j. Denote its elements by D_{lj}. The equations corresponding to the eqs. (A 6, ..., A10) are now quite simple

$$\begin{aligned}
&D_{lj}\text{ real and }\geqslant 0\\
&(2-i\bar{a}+l+j)\,D_{l+1\,j+1}=(2-ia+l+j)\,D_{lj}.\\
&(1-i\bar{a}+l-j)\,D_{l+1\,j-1}=(1-ia+l-j)\,D_{lj}.
\end{aligned} \tag{30}$$

We notice that $J_+=l+j$ and $J_-=l-j$ are changed in steps of order two so that the irreducible spaces are first of all characterized by J_+ mod. 2 and J_- mod. 2. Fig. 2 shows the (l, j)-values of the eight different subspaces.

By taking real and imaginary parts of eqs. (30) we get

$$\begin{aligned}
(2+l+j-a_2)\,D_{l+1\,j+1}&=(2+l+j+a_2)\,D_{lj}\\
(1+l-j-a_2)\,D_{l+1\,j-1}&=(1+l-j+a_2)\,D_{lj}\\
a_1\,D_{l+1\,j+1}&=a_1\,D_{lj}\\
a_1\,D_{l+1\,j-1}&=a_1\,D_{lj}.
\end{aligned} \tag{31}$$

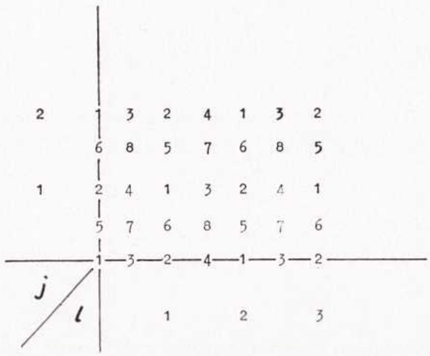

Fig. 2

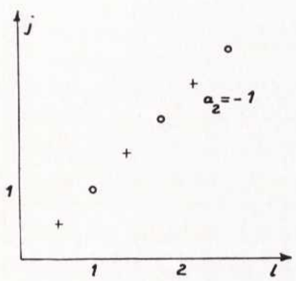

Fig. 3

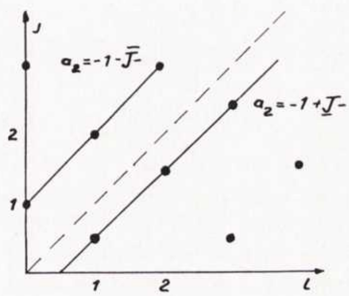

Fig. 4

If $a_1 \neq 0$ then it follows from these equations that $a_2 = 0$. Then also D_{lj} is independent of l and j and the (l, j)-contents of an irreducible representation are given by any of the subspaces in fig. 2. If $a_1 = 0$ these subspaces may still be irreducible provided a_2 is sufficiently small. On H_1 for instance $|a_2| < 1$ guarantees that $D_{l+1\,j-1}$ has the same sign as D_{lj}. On the other subspaces H_i the interval is smaller. The discrete series are obtained as follows. For $a_2 = -1$ D_{lj} can be different from zero only if $l = j$. In this way we obtain two representations with the (l, j)-contents of fig. 3.

Furthermore, J_- can have an upper bound $\bar{J}_- < 0$ or a lower bound $\underline{J}_- > 0$. This requires

$$a_2 = -1 - \bar{J}_-$$

and

$$a_2 = -1 + \underline{J}_- \tag{32}$$

respectively. These series are illustrated in fig. 4.

We have summarized some properties of the different series of representations in the tables 2 and 3. It is clear that all our representations are irreducible because reducibility would mean that there must exist other solutions for the matrix B_{ljnk} or D_{lk} which project out a space of lower dimension. However, by construction the solutions for B and D are minimal in the obove sense.

Table 2. Irreducible unitary representations characterized by real values of a and complex values of b and c ($e_1 = \mathrm{Re}\,(b+c)$, $e_2 = \mathrm{Im}\,(b+c)$, $d_1 = \mathrm{Re}\,(b-c)$, $d_2 = \mathrm{Im}\,(b-c)$).

Representation space	Series of representations characterized by the ranges of the parameters and restrictions on $n+k$ and $n-k$						
	Masin series, b and c real. Local scalar product, No restrictions on $n+k$ or $n-k$	Continuous supplementary series. Non-local scalar product. No restrictions on $n+k$ or $n-k$			Discrete series. Non-local scalar product. Bounds on $n+k$ or $n-k$ or both		
		$e_1=0$ $d_2=0$	$e_2=0$ $d_1=0$	$e_1=0$ $d_1=0$	$e_1=0$, e_2 discrete $b-c$ continuous	$b+c$ continuous $d_1=0$, d_2 discrete	$e_1=0$, e_2 discrete $d_1=0$, d_2 discrete
H_{00}: $n+k=0, \pm 2, \pm 4, \ldots$ $n-k=0, \pm 2, \pm 4, \ldots$	$-\infty < e_1 < \infty$ $-\infty < d_1 < \infty$	$0<\lvert e_2\rvert<1$ $-\infty < d_1 < \infty$	$-\infty < e_1 < \infty$ $0<\lvert d_2\rvert<1$	$0<\lvert e_2\rvert<1$ $0<\lvert d_2\rvert<1$	$e_2 = -(\overline{n+k}+1)$; $n+k \leqslant \overline{n+k}$ or $e_2 = (\underline{n+k}-1)$; $n+k \geqslant \underline{n+k}$ or $e_2 = -1$; $n+k \equiv 0$	$d_2 = -(\overline{n-k}+1)$; $n-k \leqslant \overline{n-k}$ or $d_2 = (\underline{n-k}-1)$; $n-\mathrm{k} \geqslant \underline{n-\mathrm{k}}$ or $d_2 = -1$; $n-k \equiv 0$	$e_2 - (\overline{n+k}+1)$ ⟷ $d_2 = -(\overline{n-k}+1)$ $e_2 = (\underline{n+k}-1)$ ⟷ $d_2 = (\underline{n-k}-1)$ $e_2 = -1$ ⟷ $d_2 = -1$ (The arrow denotes "combines with") $n+k \leqslant \overline{n+k}$, $n+k \geqslant \underline{n+k}$, $n-k \leqslant \overline{n-k}$, $n-k \geqslant \underline{n-k}$
H_{01}: $n+k=0, \pm 2, \pm 4, \ldots$ $n-k=\pm 1, \pm 3, \quad \ldots$	$-\infty < e_1 < \infty$ $0<\lvert d_1\rvert<\infty$	$0<\lvert e_2\rvert<1$ $0<\lvert d_1\rvert<\infty$	empty	empty			
H_{10}: $n+k=\pm 1, \pm 3, \quad \ldots$ $n-k=0, \pm 2, \pm 4, \ldots$	$0<\lvert e_1\rvert<\infty$ $0<\lvert d_1\rvert<\infty$	empty	$0<\lvert e_1\rvert<\infty$ $0<\lvert d_2\rvert<1$	empty			
H_{11}: $n+k=\pm 1, \pm 3. \quad \ldots$ $n-k=\pm 1, \pm 3, \quad \ldots$	$0<\lvert e_1\rvert<\infty$ $0<\lvert d_1\rvert<\infty$	empty	empty	empty			
$H_{\frac12\frac12}$: $n+k=\ldots -\frac32, \frac12, \frac52, \ldots$ $n-k=\ldots -\frac32, \frac12, \frac52, \ldots$	$-\infty < e_1 < \infty$ $-\infty < d_1 < \infty$	$0<\lvert e_2\rvert<\frac12$ $-\infty < d_1 < \infty$	$-\infty < e_1 < \infty$ $0<\lvert d_2\rvert<\frac12$	$0<\lvert e_2\rvert<\frac12$ $0<\lvert d_2\rvert<\frac12$			
$H_{\frac12\frac32}$: $n+k=\ldots -\frac32, \frac12, \frac52, \ldots$ $n-k=\ldots -\frac12, \frac32, \frac72, \ldots$	$-\infty < e_1 < \infty$ $-\infty < d_1 < \infty$	$0<\lvert e_2\rvert<\frac12$ $-\infty < d_1 < \infty$	$-\infty < e_1 < \infty$ $0<\lvert d_2\rvert<\frac12$	$0<\lvert e_2\rvert<\frac12$ $0<\lvert d_2\rvert<\frac12$			
$H_{\frac32\frac12}$: $n+k=\ldots -\frac12, \frac32, \frac72, \ldots$ $n-k=\ldots -\frac32, \frac12, \frac52, \ldots$	$-\infty < e_1 < \infty$ $-\infty < d_1 < \infty$	$0<\lvert e_2\rvert<\frac12$ $-\infty < d_1 < \infty$	$-\infty < e_1 < \infty$ $0<\lvert d_2\rvert<\frac12$	$0<\lvert e_2\rvert<\frac12$ $0<\lvert d_2\rvert<\frac12$			
$H_{\frac32\frac32}$: $n+k-\ldots -\frac12, \frac32, \frac72, \ldots$ $n-k=\ldots -\frac12, \frac32, \frac72, \ldots$	$-\infty < e_1 < \infty$ $-\infty < d_1 < \infty$	$0<\lvert e_2\rvert<\frac12$ $-\infty < d_1 < \infty$	$-\infty < e_1 < \infty$ $0<\lvert d_2\rvert<\frac12$	$0<\lvert e_2\rvert<\frac12$ $0<\lvert d_2\rvert<\frac12$			

Table 3. Degenerate series of representations characterized by $b=i$, $c=0$.

	Series of representations characterized by the ranges of a_1 and a_2 and restrictions on $1+j$ and $1-j$				
	Continuous series		Discrete series		
Representation space	$a_2=0$	$a_1=0$	$a_1=0$		$a_1=0$
H_1: $l+j=0, 2, 4, \dots$ $l-j=0, \pm 2, \pm 4, \dots$	$-\infty<a_1<\infty$	$0<\lvert a_2\rvert<1$	$a_2=1, 3, \dots$ $l-j\leqslant -1-a_2$ or $l-j\geqslant 1+a_2$		$a_2=-1$ $l-j\equiv 0$
H_2: $l+j=1, 3, 5, \dots$ $l-j=\pm 1, \pm 3, \dots$	$0<\lvert a_1\rvert<\infty$	empty	$a_2=0, 2, \dots$ $l-j\leqslant -1-a_2$ or $l-j\geqslant 1+a_2$		empty
H_3: $l+j=\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \dots$ $l-j=\dots -\frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \dots$	$-\infty<a_1<\infty$	$0<\lvert a_2\rvert<\frac{1}{2}$	$a_2=-\frac{1}{2}, \frac{3}{2}, \dots$ $l-j\leqslant -1-a_2$	$a_2=-\frac{1}{2}, \frac{3}{2}, \dots$ $l-j\geqslant 1-a_2$	empty
H_4: $l+j=\frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \dots$ $l-j=\dots -\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \dots$	$-\infty<a_1<\infty$	$0<\lvert a_2\rvert<\frac{1}{2}$	$a_2=-\frac{1}{2}, \frac{3}{2}, \dots$ $l-j\leqslant -1-a_2$	$a_2=\frac{1}{2}, \frac{5}{2}, \dots$ $l-j\geqslant 1+a_2$	empty
H_5: $l+j=\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \dots$ $l-j=\dots -\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \dots$	$-\infty<a_1<\infty$	$0<\lvert a_2\rvert<\frac{1}{2}$	$a_2=-\frac{1}{2}, \frac{3}{2}, \dots$ $l-j\leqslant -1-a_2$	$a_2=\frac{1}{2}, \frac{5}{2}, \dots$ $l-j\geqslant 1+a_2$	empty
H_6: $l+j=\frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \dots$ $l-j=\dots -\frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \dots$	$-\infty<a_1<\infty$	$0<\lvert a_2\rvert<\frac{1}{2}$	$a_2=\frac{1}{2}, \frac{5}{2}, \dots$ $l-j\leqslant -1-a_2$	$a_2=-\frac{1}{2}, \frac{5}{2}, \dots$ $l-j\geqslant 1+a_2$	empty
H_7: $l+j=1, 3, 5, \dots$ $l-j=0, \pm 2, \pm 4, \dots$	$-\infty<a_1<\infty$	$0<\lvert a_2\rvert<1$	$a_2=1, 3, \dots$ $l-j\leqslant -1-a_2$ or $l-j\geqslant 1+a_2$		$a_2=-1$ $l-j\equiv 0$
H_8: $l+j=2, 4, 6, \dots$ $l-j=\pm 1, \pm 3, \dots$	$0<\lvert a_1\rvert<\infty$	empty	$a_2=0, 2, \dots$ $l-j\leqslant -1-a_2$ or $l-j\geqslant 1+a_2$		empty

In addition to the series of the tables 2 and 3 there are certainly others which require a non-diagonal $\mathcal{M}$-matrix. It seems to be very difficult to solve the general recursion relations for the $\mathcal{M}$-matrix. Some more series can be obtained by varying the parametrization (9). Then one would get new equations for the $\mathcal{M}$-matrix and the diagonal solutions do not coincide with those we have calculated. However, since one does not reach all representations even after this modification and since we already have a large number of representations which may lend themselves to physical interpretation we have not considered it worth while to carry out this program at the present stage.

Finally, we want to stress that we have not examined the unitary equivalence of the representations of the tables 2 and 3. Some information can be obtained from the values of the invariants. When the invariants take on different values then the corresponding representations are clearly inequivalent. The same is true when the (l, j) contents of two representations are different. However, the question of unitary equivalence is more a mathematical problem than a physical one.

CERN, Geneva, and Institute of theoretical Physics, Göteborg, Sweden. February 1966

APPENDIX

The functions

$$T^l_{mn}(\varphi, \theta, \psi), \qquad 0 \leqslant \theta \leqslant \pi$$
$$2l = 0, 1, 2, \ldots, \qquad 0 \leqslant \varphi + \psi \leqslant 4\pi$$
$$-l \leqslant m, n \leqslant l \qquad -2\pi \leqslant \varphi - \psi \leqslant 2\pi$$

have the following properties

$$T^l_{mn}(\varphi, \theta, \psi) = e^{im\varphi} P^l_{mn}(c\theta)\, e^{in\psi},$$

where
$$P^l_{mn}(c\theta) = \frac{(-1)^{l-n}}{2^l (l-n)!} \sqrt{\frac{(l-n)!\,(l+m)!}{(l+n)!\,(l-m)!}}. \tag{A1}$$

$$(1-c\theta)^{-(m-n)/2}(1+c\theta)^{-(m+n)/2} \frac{d^{l-m}}{d(c\theta)^{l-m}} [(1-c\theta)^{l-n}(1+c\theta)^{l+n}].$$

By introducing the matrix

$$C^{lm} = \begin{bmatrix} \sqrt{\dfrac{(l-m)(l-m+1)}{(2l+1)(2l+2)}} & \sqrt{\dfrac{(l+m+1)(l-m)}{2l(l+1)}} & \sqrt{\dfrac{(l+m)(l+m+1)}{2l(2l+1)}} \\ \sqrt{\dfrac{(l+m+1)(l-m+1)}{(2l+1)(l+1)}} & \dfrac{m}{\sqrt{l(l+1)}} & -\sqrt{\dfrac{(l+m)(l-m)}{l(2l+1)}} \\ \sqrt{\dfrac{(l+m)(l+m+1)}{(2l+1)(2l+2)}} & -\sqrt{\dfrac{(l+m)(l-m+1)}{2l(l+1)}} & \sqrt{\dfrac{(l-m)(l-m+1)}{2l(2l+1)}} \end{bmatrix}$$

one has the following relations

$$\begin{aligned}
s\theta \frac{d}{d\theta} T^l_{m,n} &= (n - mc\theta) T^l_{m,n} - \sqrt{(l+m)(l-m+1)}\, s\theta e^{i\varphi} T^l_{m-1,n} \\
c\theta\, T^l_{m,n} &= C^{lm}_{21} C^{ln}_{21} T^{l+1}_{m,n} + C^{lm}_{22} C^{ln}_{22} T^l_{m,n} + C^{lm}_{23} C^{ln}_{23} T^{l-1}_{m,n} \\
-\frac{s\theta}{\sqrt{2}} e^{i\varphi} T^l_{m-l,n} &= C^{lm}_{31} C^{ln}_{21} T^{l+1}_{m,n} + C^{lm}_{32} C^{ln}_{22} T^l_{m,n} + C^{lm}_{33} C^{ln}_{23} T^{l-1}_{m,n} \\
\frac{s\theta}{\sqrt{2}} \bar{e}^{i\varphi} T^l_{m,n-1} &= C^{lm}_{11} C^{ln}_{21} T^{l+1}_{m,n} + C^{lm}_{12} C^{ln}_{22} T^l_{m,n} + C^{lm}_{13} C^{ln}_{23} T^{l-1}_{m,n} \\
\frac{s\theta}{\sqrt{2}} e^{i\psi} T^l_{m,n-1} &= C^{lm}_{21} C^{ln}_{31} T^{l+1}_{m,n} + C^{lm}_{22} C^{ln}_{32} T^l_{m,n} + C^{lm}_{23} C^{ln}_{33} T^{l-1}_{m,n} \\
\frac{s\theta}{\sqrt{2}} \bar{e}^{i\psi} T^l_{m,n+1} &= C^{lm}_{21} C^{ln}_{11} T^{l+1}_{m,n} + C^{lm}_{22} C^{ln}_{12} T^l_{m,n} + C^{lm}_{23} C^{ln}_{13} T^{l-1}_{m,n}.
\end{aligned} \tag{A2}$$

The functions T^l_{mn} satisfy the orthogonality relations

$$\int_0^{\pi}\int_0^{4\pi}\int_{-2\pi}^{2\pi} s\theta d\theta d(\varphi+\psi)\, d(\varphi-\psi)\, \bar{T}^{l_1}_{m_1 n_1}(\varphi,\theta,\psi)\, T^{l_2}_{m_2 n_2}(\varphi,\theta,\psi) = \frac{16\,\pi^2}{2l_1+1}\,\delta_{l_1 l_2}\cdot\delta_{m_1 m_2}\cdot\delta_{n_1 n_2}. \tag{A3}$$

The action of the operators $L_{\mu\nu}$ on the basis functions $T^l_{mn}\,D^j_{nk}$ is given by

$$\begin{aligned}
L_{12}\,T^l_{m,n}\,D^j_{h,k} &= -im\,T^l_{m,n}\,D^j_{h,k}\\
L_{13}\,T^l_{m,n}\,D^j_{h,k} &= \tfrac{1}{2}\Big\{\sqrt{(l+m+1)(l-m)}\,T^l_{m+1,n}\\
&\quad -\sqrt{(l+m)(l-m+1)}\,T^l_{m-1,n}\Big\}\,D^j_{h,k}\\
L_{23}\,T^l_{m,n}\,D^j_{h,k} &= \frac{1}{2i}\Big\{\sqrt{(l+m+1)(l-m)}\,T^l_{m+1,n}\\
&\quad +\sqrt{(l+m)(l-m+1)}\,T^l_{m-1,n}\Big\}\,D^j_{h,k}\\
L_{56}\,T^l_{m,n}\,D^j_{h,k} &= -ih\,T^l_{m,k}\,D^j_{h,k}\\
L_{46}\,T^l_{m,n}\,D^j_{h,k} &= \tfrac{1}{2}\,T^l_{m,n}\Big\{\sqrt{(j+h+1)(j-h)}\,D^j_{h+1,k}\\
&\quad -\sqrt{(j+h)(j-h+1)}\,D^j_{h-1,k}\Big\}\\
L_{45}\,T^l_{m,n}\,D^j_{h,k} &= \frac{1}{2i}\,T^l_{m,n}\Big\{\sqrt{(j+h+1)(j-h)}\,D^j_{h+1,k}\\
&\quad +\sqrt{(j+h)(j-h+1)}\,D^j_{h-1,k}\Big\}\\
L_{34}\,T^l_{m,n}\,D^j_{h,k} &= +(2+ia+l+j)\,C^{lm}_{21}C^{jh}_{21}C^{ln}_{21}C^{jk}_{21}\,T^{l+1}_{m,n}\,D^{j+1}_{h,k}\\
&\quad +(2+ia+l+1)\,C^{lm}_{21}C^{jh}_{22}C^{ln}_{21}C^{jk}_{22}\,T^{l+1}_{m,n}\,D^{j}_{h,k}\\
&\quad +(2+ia+l-j-1)\,C^{lm}_{21}C^{jh}_{23}C^{ln}_{21}C^{jk}_{23}\,T^{l+1}_{m,n}\,D^{j-1}_{h,k}\\
&\quad +(2+ia+j-1)\,C^{lm}_{22}C^{jh}_{21}C^{ln}_{22}C^{jk}_{21}\,T^{l}_{m,n}\,D^{j+1}_{h,k}\\
&\quad +(2+ia+2)\,C^{lm}_{22}C^{jh}_{22}C^{ln}_{22}C^{jk}_{22}\,T^{l}_{m,n}\,D^{j}_{h,k}\\
&\quad +(2+ia-j-2)\,C^{lm}_{22}C^{jh}_{23}C^{ln}_{22}C^{jk}_{23}\,T^{l}_{m,n}\,D^{j-1}_{h,k}\\
&\quad +(2+ia-l+j-1)\,C^{lm}_{23}C^{jh}_{21}C^{ln}_{23}C^{jk}_{21}\,T^{l-1}_{m,n}\,D^{j+1}_{h,k}\\
&\quad +(2+ia-l-2)\,C^{lm}_{23}C^{jh}_{22}C^{ln}_{23}C^{jk}_{22}\,T^{l-1}_{m,n}\,D^{j}_{h,k}\\
&\quad +(2+ia-l-j-2)\,C^{lm}_{23}C^{jh}_{23}C^{ln}_{23}C^{jk}_{23}\,T^{l-1}_{m,n}\,D^{j-1}_{h,k}\\
&\quad +\tfrac{1}{4}(-n-k-1-ib-ic)\,s\theta e^{i\varphi}\,s\beta e^{i\gamma}\,T^l_{m,n}\,D^j_{h,k}\\
&\quad +\tfrac{1}{4}(n-k+1+ib-ic)\,s\theta e^{i\varphi}\,s\beta \bar{e}^{i\gamma}\,T^l_{m,n}\,D^j_{h,k}\\
&\quad +\tfrac{1}{4}(-n+k+1+ib-ic)\,s\theta \bar{e}^{i\varphi}\,s\beta e^{i\gamma}\,T^l_{m,n}\,D^j_{h,k}\\
&\quad +\tfrac{1}{4}(n+k-1-ib-ic)\,s\theta \bar{e}^{i\varphi}\,s\beta \bar{e}^{i\gamma}\,T^l_{m,n}\,D^j_{h,k}.
\end{aligned} \tag{A4}$$

A unitary representation of $\bar{L}(3,3)$ is characterized by the matrix $\mathcal{M}_{lj}(nk\,|\,n'k')$ which satisfies

$$\bar{\mathcal{M}}_{lj}(nk\,|\,n'k') = \mathcal{M}_{lj}(n'k'\,|\,nk), \tag{A5}$$

$$\begin{aligned}
&(2 - i\bar{a} + l + j\sqrt{(l+n+1)(l-n+1)(j+k+1)(j-k+1)}\,\mathcal{M}_{l+1\,j+1}(nk|n'\,k')\\
&(-2 + ia - l - j)\sqrt{(l+n'+1)(l-n'+1)(j+k'+1)(j-k'+1)}\,\mathcal{M}_{lj}(nk\,|\,n'k')\\
&+\tfrac{1}{4}(-n-k-1+i\bar{b}+i\bar{c})\sqrt{(l+n+1)(l+n+2)(j+k+1)(j+k+2)}\\
&\qquad\qquad\times\mathcal{M}_{l+1\,j+1}(n+1\,k+1\,|\,n'\,k')\\
&+\tfrac{1}{4}(n'+k'-1-ib-ic)\sqrt{(l+n')(l+n'+1)(j+k')(j+k'+1)}\,\mathcal{M}_{lj}(nk\,|\,n'-1\,k'-1)\\
&\tfrac{1}{4}(-n'-k'-1-ib-ic)\sqrt{(l-n')(l-n'+1)(j-k')(j-k'+1)}\,\mathcal{M}_{lj}(nk\,|\,n'+1\,k'+1)\\
&+\tfrac{1}{4}(n+k-1+i\bar{b}+i\bar{c})\sqrt{(l-n+1)(l-n+2)(j-k+1)(j-k+2)}\\
&\qquad\qquad\times\mathcal{M}_{l+1\,j+1}(n-1\,k-1\,|\,n'\,k')\\
&-\tfrac{1}{4}(n-k+1-i\bar{b}+i\bar{c})\sqrt{(l+n+1)(l+n+2)(j-k+1)(j-k+2)}\\
&\qquad\qquad\times\mathcal{M}_{l+1\,j+1}(n+1\,k-1\,|\,n'k')\\
&-\tfrac{1}{4}(-n'+k'+1+ib-ic)\sqrt{(l+n')(l+n'+1)(j-k')(j-k'+1)}\\
&\qquad\qquad\times\mathcal{M}_{lj}(nk\,|\,n'-1\,k'+1)\\
&-\tfrac{1}{4}(n'-k'+1+ib-ic)\sqrt{(l-n')(l-n'+1)(j+k')(j+k'+1)}\,\mathcal{M}_{lj}(nk\,|\,n'+1\,k'-1)\\
&-\tfrac{1}{4}(-n+k+1-i\bar{b}+i\bar{c})\sqrt{(l-n+1)(l-n+2)(j+k+1)(j+k+2)}\\
&\qquad\qquad\times\mathcal{M}_{l+1\,j+1}(n-1\,k+1\,|\,n'k')\\
&=0,
\end{aligned} \tag{A 6}$$

$$\begin{aligned}
&(-i\bar{a}+l+1)\sqrt{(l+n+1)(l-n+1)}\cdot k\cdot\mathcal{M}_{l+1\,j}(nk|n'\,k')\\
&+(ia-l-1)\sqrt{(l+n'+1)(l-n'+1)}\,k'\cdot\mathcal{M}_{lj}(nk|n'\,k')\\
&-\tfrac{1}{4}(-n-k-1+i\bar{b}+i\bar{c})\sqrt{(l+n+1)(l+n+2)(j-k)(j+k+1)}\\
&\qquad\qquad\times\mathcal{M}_{l+1\,j}(n+1\,k+1\,|\,n'\,k')\\
&-\tfrac{1}{4}(n'+k'-1-ib-ic)\sqrt{(l+n')(l+n'+1)(j+k')(j-k'+1)}\\
&\qquad\qquad\times\mathcal{M}_{lj}(nk|n'-1\,k'-1)\\
&+\tfrac{1}{4}(-n'-k'-1-ib-ic)\sqrt{(l-n')(l-n'+1)(j+k'+1)(j-k)}\\
&\qquad\qquad\times\mathcal{M}_{lj}(nk|n'+1\,k'+1)\\
&+\tfrac{1}{4}(n+k-1+i\bar{b}+i\bar{c})\sqrt{(l-n+1)(l-n+2)(j+k)(j-k+1)}\\
&\qquad\qquad\times\mathcal{M}_{l+1\,j}(n-1\,k-1\,|\,n'\,k')
\end{aligned}$$

$$
\begin{aligned}
&-\tfrac{1}{4}(n-k+1-i\bar{b}+i\bar{c})\sqrt{(l+n+1)(l+n+2)(j+k)(j-k+1)}\\
&\qquad\qquad\times \mathfrak{M}_{l+1j}(n+1\,k-1|n'\,k')\\
&-\tfrac{1}{4}(-n'+k'+1+ib-ic)\sqrt{(l+n')(l+n'+1)(j+k'+1)(j-k')}\\
&\qquad\qquad\times \mathfrak{M}_{lj}(nk|n'-1\,k'+1)\\
&+\tfrac{1}{4}(n'-k'+1+ib-ic)\sqrt{(l-n')(l-n'+1)(j+k')(j-k'-1)}\\
&\qquad\qquad\times \mathfrak{M}_{lj}(nk|n'+1\,k'-1)\\
&+\tfrac{1}{4}(-n+k+1-i\bar{b}+i\bar{c})\sqrt{(l-n+1)(l-n+2)(j+k+1)(j-k)}\\
&\qquad\qquad\times \mathfrak{M}_{l+1j}(n-1\,k+1|n'\,k')\\
&=0,
\end{aligned}
\tag{A7}
$$

$$
\begin{aligned}
&(1-i\bar{a}+l-j)\sqrt{(l+n+1)(j-n+1)(j+k)(j-k)}\,\mathfrak{M}_{l+1j-1}(nk|n'\,k')\\
&+(-1+ia-l+j)\sqrt{(l+n'+1)(l-n'+1)(j+k')(j-k')}\,\mathfrak{M}_{lj}(nk|n'\,k')\\
&-\tfrac{1}{4}(-n-k-1+i\bar{b}+i\bar{c})\sqrt{(l+n+1)(l+n+2)(j-k-1)(j-k)}\\
&\qquad\qquad\times \mathfrak{M}_{l+1j-1}(n+1\,k+1|n'\,k')\\
&-\tfrac{1}{4}(n'+k'-1-ib-ic)\sqrt{(l+n')(l+n'+1)(j-k')(j-k'+1)}\\
&\qquad\qquad\times \mathfrak{M}_{lj}(nk|n'-1\,k'-1)\\
&-\tfrac{1}{4}(-n'-k'-1-ib-ic)\sqrt{(l-n')(l-n'+1)(j+k')(j+k'+1)}\\
&\qquad\qquad\times \mathfrak{M}_{lj}(nk|n'+1\,k'+1)\\
&-\tfrac{1}{4}(n+k-1+i\bar{b}+i\bar{c})\sqrt{(l-n+1)(l-n+2)(j+k-1)(j+k)}\\
&\qquad\qquad\times \mathfrak{M}_{l+1j-1}(n-1\,k-1|n'\,k')\\
&+\tfrac{1}{4}(n-k+1-i\bar{b}+i\bar{c})\sqrt{(l+n+1)(l+n+2)(j+k-1)(j+k)}\\
&\qquad\qquad\times \mathfrak{M}_{l+1j-1}(n+1\,k-1|n'\,k')\\
&+\tfrac{1}{4}(-n'+k'+1+ib-ic)\sqrt{(l+n')(l+n'+1)(j+k')(j+k'+1)}\\
&\qquad\qquad\times \mathfrak{M}_{lj}(nk|n'-1\,k'+1)\\
&+\tfrac{1}{4}(n'-k'+1+ib-ic)\sqrt{(l-n')(l-n'+1)(j-k')(j-k'+1)}\\
&\qquad\qquad\times \mathfrak{M}_{lj}(nk|n'+1\,k'-1)\\
&+\tfrac{1}{4}(-n+k+1-i\bar{b}+i\bar{c})\sqrt{(l-n+1)(l-n+2)(j-k-1)(j-k)}\\
&\qquad\qquad\times \mathfrak{M}_{l+1j-1}(n-1\,k+1|n'\,k')\\
&=0,
\end{aligned}
\tag{A8}
$$

$$
\begin{aligned}
&(1-i\bar{a}+j)\,n\sqrt{(j+k+1)(j-k+1)}\,\mathfrak{M}_{lj+1}(nk|n'k')\\
&+(-1+ia-j)\,n'\sqrt{(j+k'+1)(j-k'+1)}\,\mathfrak{M}_{lj}(nk|n'k')\\
&-\tfrac{1}{4}(-n-k-1+i\bar{b}+i\bar{c})\sqrt{(l+n+1)(l-n)(j+k+1)(j+k+2)}\\
&\qquad\qquad\times\mathfrak{M}_{lj+1}(n+1\,k+1|n'k')\\
&-\tfrac{1}{4}(n'+k'-1-ib-ic)\sqrt{(l+n')(l-n'+1)(j+k')(j+k'+1)}\\
&\qquad\qquad\times\mathfrak{M}_{lj}(nk|n'-1\,k'-1)\\
&+\tfrac{1}{4}(n+k-1+i\bar{b}+i\bar{c})\sqrt{(l+n)(l-n+1)(j-k+1)(j-k+2)}\\
&\qquad\qquad\times\mathfrak{M}_{lj+1}(n-1\,k-1|n'k')\\
&+\tfrac{1}{4}(-n'-k'-1-ib-ic)\sqrt{(l+n'+1)(l-n')(j-k')(j-k'+1)}\\
&\qquad\qquad\times\mathfrak{M}_{lj}(nk|n'+1\,k'+1)\\
&+\tfrac{1}{4}(n-k+1-i\bar{b}+i\bar{c})\sqrt{(l+n+1)(l-n)(j-k+1)(j-k+2)}\\
&\qquad\qquad\times\mathfrak{M}_{lj+1}(n+1\,k-1|n'k')\\
&+\tfrac{1}{4}(-n'+k'+1+ib-ic)\sqrt{(l+n')(l-n'+1)(j-k')(j-k'+1)}\\
&\qquad\qquad\times\mathfrak{M}_{lj}(nk|n'-1\,k'+1)\\
&-\tfrac{1}{4}(n'-k'+1+ib-ic)\sqrt{(l+n'+1)(l-n')(j+k')(j+k'+1)}\\
&\qquad\qquad\times\mathfrak{M}_{lj}(nk|n'+1\,k'-1)\\
&-\tfrac{1}{4}(-n+k+1-i\bar{b}+i\bar{c})\sqrt{(l+n)(l-n+1)(j+k+1)(j+k+2)}\\
&\qquad\qquad\times\mathfrak{M}_{lj+1}(n-1\,k+1|n'k')\\
&=0,
\end{aligned}
\tag{A9}
$$

$$
\begin{aligned}
&(-i\bar{a}nk+ian'k')\,\mathfrak{M}_{lj}(nk|n'k')\\
&+\tfrac{1}{4}(-n-k-1+i\bar{b}+i\bar{c})\sqrt{(l+n+1)(l-n)(j+k+1)(j-k)}\,\mathfrak{M}_{lj}(n+1\,k+1|n'k')\\
&+\tfrac{1}{4}(n'+k'-1-ib-ic)\sqrt{(l+n')(l-n'+1)(j+k')(j-k'+1)}\,\mathfrak{M}_{lj}(nk|n'-1\,k'-1)\\
&+\tfrac{1}{4}(-n'-k'-1-ib-ic)\sqrt{(l+n'+1)(l-n')(j+k'+1)(j-k')}\,\mathfrak{M}_{lj}(nk|n'+1\,k'+1)\\
&+\tfrac{1}{4}(n+k-1+i\bar{b}+i\bar{c})\sqrt{(l+n)(l-n+1)(j+k)(j-k+1)}\,\mathfrak{M}_{lj}(n-1\,k-1|n'k')\\
&+\tfrac{1}{4}(n-k+1-i\bar{b}+i\bar{c})\sqrt{(l+n+1)(l-n)(j+k)(j-k+1)}\,\mathfrak{M}_{lj}(n+1\,k-1|n'k')\\
&+\tfrac{1}{4}(-n'+k'+1+ib-ic)\sqrt{(l+n')(l-n'+1)(j+k'+1)(j-k')}\\
&\qquad\qquad\times\mathfrak{M}_{lj}(nk|n'-1\,k'+1)\\
&+\tfrac{1}{4}(n'-k'+1+ib-ic)\sqrt{(l+n'+1)(l-n')(j+k')(j-k'+1)}\,\mathfrak{M}_{lj}(nk|n'+1\,k'-1)\\
&+\tfrac{1}{4}(-n-k+1-i\bar{b}+i\bar{c})\sqrt{(l+n)(l-n+1)(j+k+1)(j-k)}\,\mathfrak{M}_{lj}(n-1\,k+1|n'k')\\
&=0.
\end{aligned}
\tag{A10}
$$

ACKNOWLEDGEMENTS

I want to thank Prof. N. Svartholm for his constant encouragement and advise concerning the presentation of this work. To Dr. J. Nilsson who has read and earlier manuscript I also owe thanks.

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Tryckt den 2 maj 1966

Uppsala 1966. Almqvist & Wiksells Boktryckeri AB

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Some non-compact symmetry groups for elementary particles associated with a geometrical model

AWE

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GÖTEBORG · UPPSALA
1966

3.66 04 5 Communicated 23 February 1966 by L. Hulthén and N. Svartholm

Some non-compact symmetry groups for elementary particles associated with a geometrical model

By Arne Kihlberg

ABSTRACT

Three Lie-groups containing the Poincaré group and one of the groups SO(2), L(1, 3) or L(3, 3) respectively, are studied with respect to their ability of reproducing the spectra of the internal quantum numbers. Their use as interaction symmetry groups is discussed.

1. Introduction

The lack of a consistent dynamical theory which can explain all processes in elementary particle physics has motivated the search for symmetries which would impose restrictions on the interactions among the particles and resonances. Some symmetries are well known and find their natural explanation in the theory of relativity. More precisely, we think of the symmetry of all physical laws under translations and rotations of the four-dimensional coordinate system. Since these symmetries have an unrestricted validity it is comforting that they have a geometrical explanation. Other experimentally observed symmetries, mostly referred to as internal symmetries, have a more restricted domain of validity. In recent years these internal symmetries have been discussed primarily within the frame-work of the group SU(3) [1]. However, the SU(3) theory is purely phenomenological and it does not seem to emerge naturally from any simple geometrical model. One might possibly argue that a basic theory of internal symmetries cannot be geometrical since the SU(3) symmetry is valid only for the very strongest interactions. Electromagnetic and weak interactions and also the weaker part of strong interactions violate this symmetry. However, there is an objection to this reasoning. A geometrical explanation may in some way be connected to, say, electromagnetism and then the symmetry transformations might well cease to be symmetry operations for the electromagnetic interactions.

One can proceed along different lines when one tries to find basic explanations for the internal symmetries which are found experimentally. Either one takes the groups SU(3) for granted and tries to find a geometrical or dynamical interpretation of it, or one may look for possible extensions of the space-time symmetry group, that is the Poincaré group P. The latter procedure might not lead to SU(3) as the internal symmetry group but rather to some other group. Presently one has some confidence in SU(3), but its capacity is not so great that one should give up the hope of finding other groups which better reflect the basic properties of Nature. Of course, one may question whether the regularities of the elementary particle interactions should necessarily be explained on the basis of a group. In the past, however, this idea has been very successful.

During the last couple of years there has been a vivid interest in the possibility of merging the internal and the external (i.e. space-time) symmetries into one large symmetry group. Most investigations have led to negative results [2], or to proposals which are hard to interpret [3]. Others seem somewhat more promising [4] and the model which has been studied most extensively is the relativistic version of the SU(6) theory [5]. In that case the global group, which contains both P and SU(3), is either ISL(6, C) [6], SU(6, 6) [7] or some even larger group. One difficulty with all these groups is that they require the introduction of an euclidean space of very high dimension which is difficult to interpret physically.

Theories based on geometrical considerations have been discussed by many authors. All those theories deal with the conventional space-time or some extension of it. Fröhlich [8] has tried to find new symmetries defined on space-time. Allcock [9] and Vigier *et al.* [10] have used a space defined by the coordinates of a rotator-like structure. In a number of papers [11] Rayski has examined the possibility of extending space-time by two extra time-like dimensions. The resulting space is thus the pseudo-euclidean space with the signature $(+++---)$. The spin is assumed to emerge from the three-dimensional rotation group acting on the space coordinates whereas isospin has a similar relation to the time coordinates. Just like the relativistic versions of the SU(6) theory it has the unattractive feature of introducing unobservable coordinates and additional assumptions must be made to prevent them from being observed.

More recently several authors [4] [12] have proposed the use of groups not for expressing symmetries but merely to account for the spectra of various quantized observables. This idea is a natural extension of the observation that the energy spectrum of the hydrogen atom can be generated by the non-compact group L(1,4) [13]. The group is then called a spectrum generating group.

In a previous paper [14] we have defined an extended coordinate space for an elementary system by utilizing both the spatial and the polarization properties of the photon. We were then led to an eight-dimensional coordinate space. Four of the coordinates span space-time, three angles define the orientation of the particle at each point in space-time and the eighth coordinate is a scale-factor.

To change the basic manifold from the usual four-dimensional Minkowski-space to an eight-dimensional manifold necessarily implies a revised theory of the elementary particles. The study of such an altered elementary particle theory could carried out in two steps

A: a study of transformation groups on the eight-dimensional manifold and their use as symmetry groups.

B: the construction of a dynamical theory, eventually in the form of a quantized field theory based on the eight-dimensional space.

With regard to point B we just want to mention that similar programs have been proposed with the aim of using the ten-dimensional group space of the Poincaré group as the basic manifold [15]. We willl devote future work to the development of point B.

The line of investigation of point A has been followed in a sequence of papers, hereafter referred to as paper I, II, ... IV [14] [16]. In paper I we introduced the eight-dimensional space and made a preliminary study of the transformation groups. Paper II demonstrated the possibility of using part of the space for an explicit realization of the unitary representations of the Lorentz group. Paper III contains a general study of the pseudo-orthogonal groups ($L(p, q)$ and their unitary representations. The purpose of that paper is to find suitable methods to study the group L(3, 3), which is one of the transformation groups of interest. The detailed study of the group L(3, 3) is made in paper IV.

The present paper contains the conclusions which can be drawn from the results of the papers I to IV. In Section 2 the definition of the eight-dimensional space is reviewed and a derivation of three possible transformation groups is outlined. These groups are interpreted as global symmetry groups containing both the Poincaré group and the internal symmetry group. In Section 3 a more detailed examination of the third group is made.

2. Three transformation groups in the extended coordinate space

To define the Minkowsky space in relativity it is necessary to have a clock, which shows local time, and a "radar" station, which can emit very sharp electromagnetic signals. With this equipment it is possible to measure three-dimensional distances and also to synchronize clocks at various points in space. To do this from one single position of the radar station the station has to be attached to a massive body so that the antenna can be turned. It is proposed that one may also utilize that information of the electromagnetic radiation which is related to its stated polarization. At each point on the path of the wave packet the polarization can be used to orientate a triad defined by, say, the electromagnetic field strength $\mathbf{e}$, the magnetic field strength $\mathbf{h}$ and the cross product $\mathbf{e} \times \mathbf{h}$. By translating and rotating the radar station it is possible to have all triads lined up in the same way so that whenever an electromagnetic pulse passes through any point in space it always has the same polarization relative to the local triad. In this way we have obtained an eight-dimensional manifold defined by the four space-time coordinates x_1, x_2, x_3 and t and the four parameters which specify the six-vector $(\mathbf{e}, \mathbf{h})$. From now on we denote by $(\mathbf{e}_0, \mathbf{h}_0)$ the six-vector which serves as the reference for any six-vector $(\mathbf{e}, \mathbf{h})$. Additional degrees of freedom are introduced by attaching such interval coordinates $(\mathbf{e}, \mathbf{h})$ to any elementary system. As independent coordinates one may choose the three Euler angles between the six-vector $(\mathbf{e}_0, \mathbf{h}_0)$ and the six-vector $(\mathbf{e}, \mathbf{h})$ and the scale coordinate $s = ln|\mathbf{e}|/|\mathbf{e}_0|$.

In the eight-dimensional space so defined one can now consider the problem of finding the appropriate symmetry group of transformations. As was noted in paper I there is no canonical way of arriving at a unique group when the space is given. Rather, we will define three such groups and discuss their possible interest for physics. Just as the Poincaré group leaves the basic time-definition invariant in relativity our extended group should also leave the parallellity of the six-vectors $(\mathbf{e}_0, \mathbf{h}_0)$ at different points unchanged. But even so we are quite free to choose our group. We start by listing the infinitesimal generators of the Poincaré group. We shall in general discuss only the infinitesimal generators of the group. This does not imply any restriction since in quantum mechanics it is usually not the original group which is of interest, but rather some covering group, and the universal covering group is closely related to the Lie-algebra. Thus, we define the generators

$$p_t = \frac{\partial}{\partial t}$$

$$p_i = \frac{\partial}{\partial x_i} \; (i = 1, 2, 3) \tag{1}$$

for time and space translation, and

$$M_i = -\varepsilon_{ijk} x_j \frac{\partial}{\partial x_k} + S_i$$

$$M'_i = t \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial t} + S'_i \tag{2}$$

for space rotations and accelerations. The accelerations and rotations clearly act both on the space-time variables $(\mathbf{x}, t)$ and on the six-vector $(\mathbf{e}_0, \mathbf{h}_0)$. The expressions for S_i and S'_i can be given in terms of the Euler angles

$$S_1 = C\varphi \frac{C\theta}{S\theta} \frac{\partial}{\partial \varphi} + S\varphi \frac{\partial}{\partial \theta} - \frac{C\varphi}{S\theta} \frac{\partial}{\partial \psi}$$

$$S_2 = S\varphi \frac{C\theta}{S\theta} \frac{\partial}{\partial \varphi} - C\varphi \frac{\partial}{\partial \theta} - \frac{S\varphi}{S\theta} \frac{\partial}{\partial \psi}$$

$$S_3 = -\frac{\partial}{\partial \varphi}$$

$$S'_1 = \frac{S\varphi}{S\theta} \frac{\partial}{\partial \varphi} - C\varphi C\theta \frac{\partial}{\partial \theta} - S\varphi \frac{C\theta}{S\theta} \frac{\partial}{\partial \psi} - C\varphi S\theta \left(\frac{\partial}{\partial s} - 1 \right)$$

$$S'_2 = -\frac{C\varphi}{S\theta} \frac{\partial}{\partial \varphi} - S\varphi C\theta \frac{\partial}{\partial \theta} + C\varphi \frac{C\theta}{S\theta} \frac{\partial}{\partial \psi} - S\varphi S\theta \left(\frac{\partial}{\partial s} - 1 \right)$$

$$S'_3 = S\theta \frac{\partial}{\partial \theta} - C\theta \left(\frac{\partial}{\partial s} - 1 \right) \tag{3}$$

For convenience there are some changes in the notations of this paper as compared to paper I. Also the operational definition of S_i and T_i given in paper I is different from the one used here and in paper IV.

Let us now introduce the following generators

$$T_1 = -\frac{C\psi}{S\theta} \frac{\partial}{\partial \varphi} + S\psi \frac{\partial}{\partial \theta} + C\psi \frac{C\theta}{S\theta} \frac{\partial}{\partial \psi}$$

$$T_2 = -\frac{S\psi}{S\theta} \frac{\partial}{\partial \varphi} - C\psi \frac{\partial}{\partial \theta} + S\psi \frac{C\theta}{S\theta} \frac{\partial}{\partial \psi} \tag{4}$$

$$T_3 = -\frac{\partial}{\partial \psi}.$$

They generate transformations of the Euler angles but now acting on the six-vector (e, h). Thus T_1, T_2 and T_3 commute with S_1, S_2 and S_3. The operator T_3 is distinguished since it commutes also with S_i'.

We shall first consider the transformation group G_1 which is characterized by the generators of the eqs. (1) and (2) and T_3. The group G_1 is thereby defined up to a discrete centre and has the structure

$$G_1 = \frac{\overline{P} \otimes \overline{SO}(2)}{Z_1} \tag{5}$$

where $\overline{P}$ is the universal covering group of the Poincaré group P, $\overline{SO}(2)$ is the universal covering group of the rotation group in two dimensions and Z_1 is some discrete centre of $\overline{P} \otimes \overline{SO}(2)$. If one does not regard the group G_1 as an abstract symmetry group but rather considers the explicit realization of it on the coordinate space then, of course, one obtains a specific group.

As a first step towards a discussion of G_1 let us determined its unitary irreducible representations. This is easily done since such a representation is the direct product of a unitary irreducible representation $U(m, s)$ of $\overline{P}$ and a unitary irreducible representation $U(b)$ of $\overline{SO}(2)$ which maps Z_1 on the unit operator. The real parameter b characterizes the one-dimensional representation of $\overline{SO}(2)$, m is the mass and s is the spin and these two quantities label the irreducible representation of the group $\overline{P}$. Let us now discuss the possible use of these representations in elementary particle physics. It is clear that if Z_1 consists only of the unit element of $\overline{P} \otimes \overline{SO}(2)$ then n is completely unrelated to m and s. In this case there is no obvious physical interpretation of b. However, if Z_1 consists of the elements

$$\{(\alpha, \beta)\} = \{(0, 0), (2\pi, 2\pi), (0, 4\pi), (2\pi, 6\pi) \ldots\}$$

where α denotes an angle of rotation in P and β an angle of rotation in SO(2), then it follows that b must be integer or half-integer and furthermore that

$$b + s = \text{integer} \tag{6}$$

This relation suggests the identification $b = B/2$ where B is the baryon number.

Returning to the explicit expressions (1), (2), (3) and (4) for the generators of the group and further assuming that the physically interesting representations should be obtained on the covering space of our eight-dimensional manifold then relation (6) follows automatically. To show this we introduce the Wigner functions $D^s_{s_3 b}(\varphi, \theta, \psi)$ as a spin basis in the rest frame. Then the eigenvalue of the generator T_3 of $\overline{SO}(2)$ is ib and b evidently fulfils eq. (6). In fact one also has

$$|b| \leqslant s \tag{7}$$

Both relations (6) and (7) are satisfied for all known particles and resonances.

Besides these two relations there is not much more that can be said about the spectra of quantum numbers. Clearly it is highly suggestive to extend G_1 to be not only a spectrum-generating group but also a symmetry group since B is rigorously conserved. But the Clebsh-Gordon decomposition of a direct product is almost trivial and would not give more restrictions on, say, scattering amplitudes than the Poincaré group and the baryon group do separately. However, in a dynamical model expressed in the form of a field theory where the fields depend on all eight coordinates the explicit realization of G_1 may possibly give more.

In paper I we also discussed a second transformation group G_2. This is obtained if one assumes that besides the transformations of the eqs. (1) and (2) one is also allowed to perform independent rotations of the six-vector ($\mathbf{e}_0$, $\mathbf{h}_0$). This means that the operators S_i in eq. (3) are also generators of the group. By working out the commutation relations it is then found that the Lie algebra is the direct sum of the Lie algebra of P and that of the Lorentz group $L(1, 3)$. Thus

$$G_2 = \frac{\overline{P} \otimes \overline{L}(1,3)}{Z_2} \tag{8}$$

This group considered as an abstract group or restricted to the special realization on the eight-dimensional space ought to be a good candidate for a symmetry group. This is so because all group operations are transformations on the external coordinate space and these transformations are in accordance with the basic definitions of the space. If G_2 is viewed as an abstract group then a unitary irreducible representation of G_2 is the direct product of $U(m, s)$ and a unitary irreducible representation $U(k_0, \nu)$ of $L(1, 3)$ [17] which maps Z_2 on the unit operator. The parameter k_0 is integer or half-integer and is equal to the j-value of the lowest SU(2) representation which occurs in $U(k_0, \nu)$. The choices for Z_2 are either only the unity,

$$\{(\alpha, \beta)\} = \{(0,0);\ (0,2\pi)\},\ \{(\alpha, \beta)\} = \{(0, 0),\ (2\pi, 0)\}$$
$$\{(\alpha, \beta)\} = \{(0,0),\ (2\pi,\ 2\pi)\},$$

or

$$\{(\alpha, \beta)\} = \{(0,0),\ (0,2\pi),\ (2\pi,\ 0),\ (2\pi,\ 2\pi)\}$$

where α again is a rotation angle in P and β is a rotation angle in $L(1, 3)$. Only the last two-element centre relates the representation ($U(m, s)$ to $U(k_0, \nu)$ and we have

$$s + k_0 = \text{integer} \tag{9}$$

As stated before the only known elementary particle quantum number which is related to the spin in this way is $B/2$. We next turn to the interpretation of the label j which fulfils $j \geqslant k_0$. No other quantum number than s seems to have this property. For this reason we do not find any interesting interpretation of G_2 as an abstract group. What about the explicit realization on the eight-dimensional space? Then P in eq. (8) acts only on the space coordinates while the transformations S_i, S_i are in $L(1, 3)$. It would certainly not be possible to identify S_1, S_2 and S_3 for a moving particle with its spin components since then the invariance under G_2 and thus under $L(1, 3)$ would mean conservation of j i.e. the spin. Under those circumstances the reactions $N^* \to N + \pi$ and $\varrho \to \pi + \pi$ would be forbidden. However, in the rest system we could identify S_1, S_2 and S_3 with the spin components and then define the spin in a moving system by means of a "physical" Lorentz transformation [18]. Therefore, let us look at the spectra of quantum numbers. A representation of G_2 is now characterized by three numbers the mass m, k_0 and the continuous parameter ν. By fixing $k_0 = B/2$ we then find that the representation of the physical Poincaré group (whose generators are given in eqs. (1) and (2)) is reducible and contains the spin values $B/2$, $B/2 + 1$, ... each value once. This group then suggests a multiplet characterized by the mass and the baryon number and containing an infinite number of spin states. Although the presently known particles and resonances seem to appear in such strings of spin multiplets, e.g. the nucleon resonances, we feel that the evidence is not very convincing. So far we have only considered G_2 in the rest system or rather just as a spectrum generating group. The question is whether G_2 taken as a symmetry group for particles in motion gives restrictions on scattering amplitudes which agree with experiments. To examine this one has to calculate Clebsch-Gordon functions for a product of two representations. This has been done by Bisiacchi and Frondsdal [19] and they find that the reaction $\varrho \to \pi + \pi$ is forbidden if ϱ belongs to a representation with $k_0 = 0$. This is so in our interpretation and we shall not examine the group G_2 further because of this negative feature and also because there is no room for the important isotopic spin quantum number t in the formalism.

In order to be able to include also the quantum numbers t and hypercharge Y one evidently has to take a group larger than G_2. Let us start again by the infinitesimal transformation of the Poincaré group, the eqs. (1) and (2). By adding the three generators of eq. (4) and taking all commutators it has been shown in paper V that one obtains a Lie algebra which is the direct sum of that of P and that of $L(3, 3)$, the pseudo-orthogonal group with the signature $(+++---)$. Thus one has

$$G_3 = \frac{\overline{P} \otimes \overline{L}(3,3)}{Z_3} \tag{10}$$

where Z_3 is some centre of $\overline{P} \otimes \overline{L}(3, 3)$. Just as in the case of G_1 or G_2 we can conceive G_3 in an abstract way i.e. independent of its realization on the eight-dimensional space and look for all its unitary irreducible representations. Then one would refer the spin to P. Now $L(3, 3)$ contains two compact $SO(3)$ groups and from their definition in terms of the generators of the eqs. (3) and (4) it is rather natural to assume that they are the rotation groups of spin and isospin in the rest frame. (Note that T_1, T_2 and T_3 operate on the internal coordinates ($\mathbf{e}$, $\mathbf{h}$) of the particle and that T_3 is distinguished in the same fashion as the third component t_3 of the isospin is distinguished by the electromagnetic interactions.) Thus one would have two spin groups. Also it is hard to associate S_1, S_2 and S_3 to any other quantized operators than just the spin. Therefore, we shall not consider G_3 in the abstract way any further but instead use the explicit realization (1) and (3) for the Poincaré group. Then the spin is referred to $L(3, 3)$ since S_i, S_i' are 6 of the 15 generators of $L(3, 3)$. For $L(3, 3)$ we shall consider both the possibility that 9 of its generators are given explicitly by the eqs. (3) and (4) and also the case when one looks for general representations of $L(3, 3)$. As soon as the spin is referred to the group $L(3, 3)$ it is clear that one cannot identify one of the $SO(3)$ subgroup with the spin group and still claim that G_3 is an invariance group. Just as in the case of G_2 one would then have spin conservation forbidding well-known reactions. The situation may perhaps be saved by defining the spin operators in the rest system and then applying a Lorentz transformation so that for the particle in motion the spin operators are in the enveloping algebra rather than in the Lie algebra. In any case an examination of this kind has to be preceded by a calculation of the Clebsch-Gordon series for $L(3, 3)$. Irrespective of this point we can always consider the group G_3 as a spectrum generating group and see if the experimentally observed spectra can be convincingly interpreted in terms of irreducible representations. This will be the subject of the next section. Let us also point out that although we have a group with the signature $(+++---)$ we do not introduce a six-dimensional euclidean space as Rayski does. Thus we need not explain why the additional time coordinates cannot be observed.

3. Investigation of $G_3' = P \cdot L(3,3)$ as a spectrum generating group

In this paragraph we shall examine in detail the group

$$G_3' = P \cdot \overline{L}(3,3) \tag{11}$$

where the dot means semi-direct product. We also have

$$G_3' = P' \otimes L(3,3) \tag{12}$$

but then P' contains only the spatial part i.e. the transformations on x_1, x_2, x_3 and t according to the eqs. (1) and (2). Having specified this we can forget about P', it defines just the mass and the momentum of the particle. Furthermore, we consider $L(3,3)$ as a spectrum generating group and identify the maximal compact subgroup

$$K = SO(3) \otimes SO(3) \tag{13}$$

with the product of the spin group and the isospin group.

As was shown in paper IV a vector in an irreducible representation space must be labelled by six indices denoted (l, m, n, j, h, k). The numbers l and j characterize an irreducible unitary representation of K while m and h are the "third components" of l and j, respectively. The labels n and k are a sort of degeneracy indices and they always fulfil

$$n = -l, -l+1, \ldots, l, \tag{14}$$
$$k = -j, j+1, \ldots, j$$

Paper IV contains the derivation of several series, both degenerate and non-degenerate, of unitary, irreducible representations. Each series is characterized by certain restrictions on the labels l, j, n, and k. In the non-degenerate series the restrictions are on $n+k$ and $n-k$ while l and j range from 0 to ∞ or from $\frac{1}{2}$ to ∞ in integer steps. In the degenerate series the labels n and k are absent and thus a vector in the Hilbert space has only 4 labels. Here the different series are characterized by bounds on $l-j$. The reader is referred to paper IV for a more detailed description.

Let us now turn to the physical interpretation. We recall that G_3' is to be considered as a spectrum generating group. Now the quantum numbers in elementary particle physics which correspond to operators which have non-trivial spectra are, besides the mass the spin s and its third component s_3, the baryon number B, the isospin t and its third component t_3 and, finally, the hypercharge Y. The operational definition of the group strongly indicates that we should make the identifications

$$s \equiv l$$
$$s_3 \equiv m$$
$$t \equiv j \tag{15}$$
$$t_3 \equiv h$$

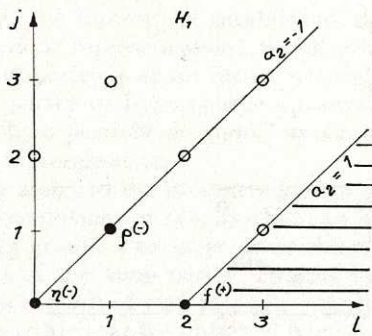

Fig. 1

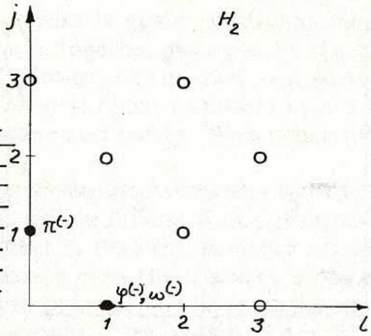

Fig. 2

For n and k we have no such indications. However, it is well known that $Y/2$ is integer when t is integer and half-integer when t is half-integer. We have noted the same connection between $B/2$ and s earlier. Therefore one could tentatively put

$$\begin{aligned} B/2 &\equiv n \\ Y/2 &\equiv k \end{aligned} \tag{16}$$

However, because of the limitations (14) such an identification is not in accordance with the assignment of hypercharge and isospin of Ω^-. But it should be remarked that this assignment is based solely on SU(3) and not on experimental facts. Turning to the series of non-degenerate representations one finds that some of these are characterized by upper or lower bounds on $n+k$ or $n-k$. If one makes a plot of B and Y of the presently known particles and resonances one finds, however, no indication of upper or lower bounds on $B \pm Y$. We have also tried to construct a mass formula without any success. Therefore, we are inclined to abandon the identification (16). This means that we give up the goal of finding the spectra of all quantum numbers and instead concentrate on the spin and isospin spectra. One may even say that this is a natural restriction since Y was introduced first in connection with strong interactions while our space is based on the properties of the photon. Also one has then free access to the degenerate representations of paper IV. Some of these are particularly interesting since they are characterized by bounds on $l-j$.

The degenerate representations are realized in eight different subspaces $H_1, \dots H_8$ characterized by $l+j$ mod 2 and $l-j$ mod 2. Let us therefore put the presently known particles and resonances at their places in the spaces H_i. We use only those resonances for which the spin and isospin values are known with reasonable certainty [20]. The space parity is also indicated in the figs. The subspaces H_5 and H_6 shall not be considered in the following discussion since they contain only one particle each. Also the non-strange mesons are not very abundant. However, it seems natural to put η and ϱ in the representation $a_2=-1, l\equiv j$ of Table 3 in paper IV. This representation is indicated in fig. 1 by a line. Note that η and ϱ have the same space parity. In the subspace H_2 we do not suggest any specific representation. For the baryons and baryon resonances we have a much larger material. In subspace H_3 we propose a representation $a_2=-\frac{1}{2}$, $l-j\geqslant\frac{1}{2}$ which will then accommodate the four particles $\Lambda(1115)$, $Y_0(1815)$, $Y_1(1383)$ and $Y_1(2065)$ all with the same hypercharge and parity.

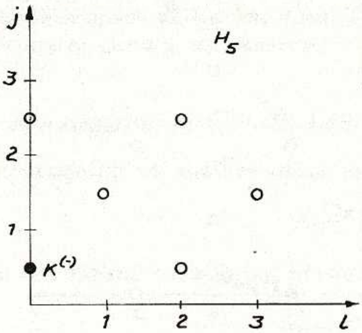

Fig. 3

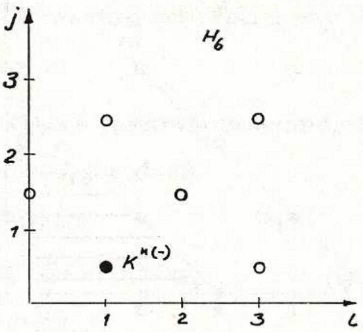

Fig. 4

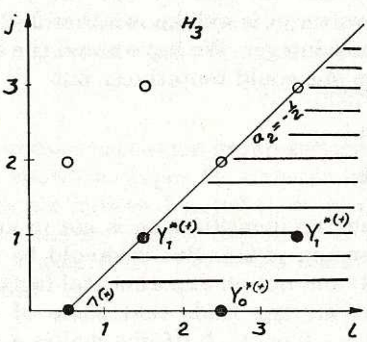

Fig. 5

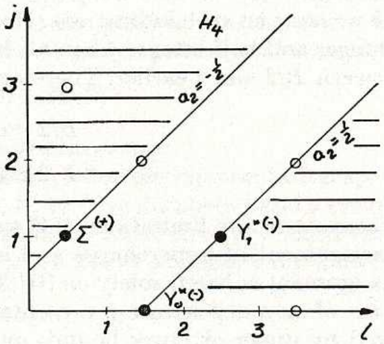

Fig. 6

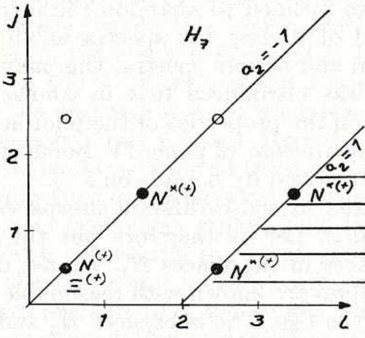

Fig. 7

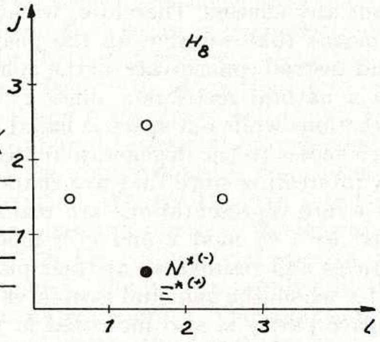

Fig. 8

Similarly all particles in subspace H_7 have parity $+1$. Despite this we propose to put the nucleon and the nucleon resonance $N^*(1236)$ in a representation $a_2=-1$, $l\equiv j$, and the other two nucleon resonances $N^*(1688)$, $N^*(1924)$ in a representation $a_2=1$, $l-j\geqslant 2$. The Ξ-particle at the same place as the nucleon must belong to a third representation since there can be only one state at each point. In the subspaces

H_4 and H_8 the number of established particles is again somewhat meagre but an assignment of representations which groups together particles of the same hypercharge and parity is again rather natural. Notice that as soon as a particle with the "wrong" parity or hypercharge appears then it always appears at a point already occupied by a particle of "right" hypercharge and parity. Thus it has to belong to a different representation.

Having assigned the presently known particles and resonances to different degenerate representations of $L(3, 3)$ what can be said in favour of this group as a spectrum generating group? The most astonishing fact is that the particles are grouped into multiplets of the same parity. For the mesons even the G-parity is the same. From a theoretical point of view it is interesting to note that we have representations of the type $1 \equiv j$ (which can be obtained from the explicit realization of $L(3, 3)$ on the four-dimensional internal space) giving rise to strings of resonances

$$P_{00}, P_{11}, P_{22}, \ldots$$

$$P_{1/2\,1/2}, P_{3/2\,3/2}, P_{5/2\,5/2} \ldots$$

Such strings are indicated in bootstrap calculations. Also one sees that an irreducible representation contains for a given isospin only spin values differing by 2 units in accordance with the Regge recurrency. It is perhaps justified to conclude from this that there are some rather strong indications that the group $L(3, 3)$ could be a spectrum generating group. We feel, however, that one has to wait for the discovery of more resonances before anything conclusive can be said.

4. Concluding remarks

In this paper three groups G_1, G_2 and G_3' have been examined as spectrum generating groups. Especially the last group G_3' shows some interesting features. The groups G_1 and G_2 only allow for the introduction of the baryon number. G_1 can certainly be conceived as an invariance group of interactions while this is not so probable for G_2. If new experimental data tend to support our proposed classification it may be worth while to examine G_3' also as an invariance group.

One should also remark that the negative conclusions concerning the possibility of identifying $B/2$ and $Y/2$ with n and k may be a consequence of non-completeness of the table 2 of non-degenerate representations in paper IV.

ACKNOWLEDGEMENT

It is a pleasure to thank Professor N. Svartholm and Laborator J. Nilsson for several constructive remarks in the course of writing this paper.

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Tryckt den 2 maj 1966

Uppsala 1966. Almqvist & Wiksells Boktryckeri AB

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